[M]athematical Tables
and other
[A]ids to Computation

A Quarterly Journal edited on behalf of the
Committee on Mathematical Tables
and Other Aids to Computation
by

RAYMOND CLARE ARCHIBALD
DERRICK HENRY LEHMER
WITH THE COÖPERATION OF
LESLIE JOHN COMRIE
SOLOMON ACHILLOVICH JOFFE

II · Number 20 · October, 1947

Published by
THE NATIONAL RESEARCH COUNCIL

NATIONAL RESEARCH COUNCIL
Division of Physical Sciences

COMMITTEE ON MATHEMATICAL TABLES AND OTHER AIDS TO COMPUTATION

Published quarterly in January, April, July and October by the National Research Council, Prince and Lemon Sts., Lancaster, Pa., and Washington, D. C.

All contributions intended for publication in *Mathematical Tables and Other Aids to Computation*, and all Books for review, should be addressed to Professor R. C. Archibald, Brown University, Providence 12, R. I.

Entered as second-class matter July 29, 1943, at the post office at Lancaster, Pennsylvania, under the Act of August 24, 1912.

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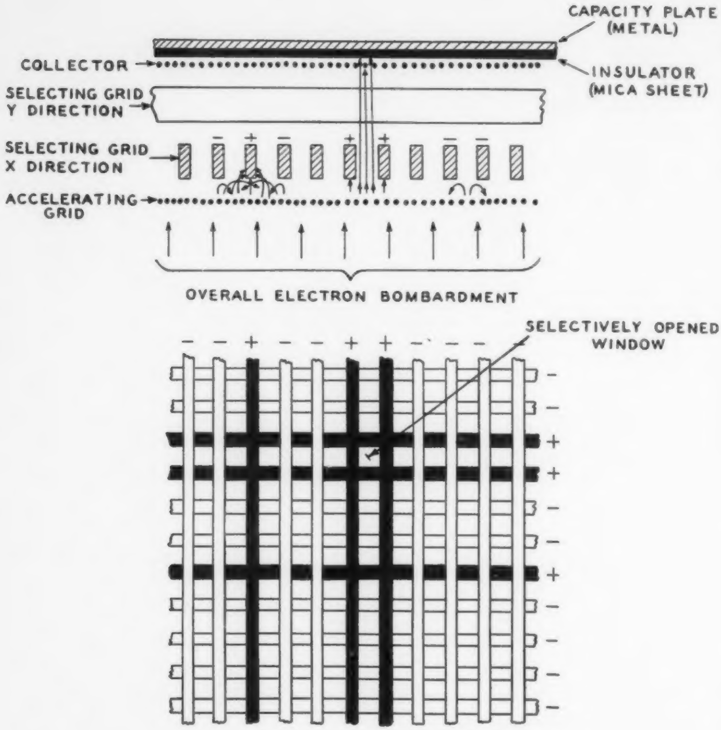

FIG. 1. Principle of selectron.

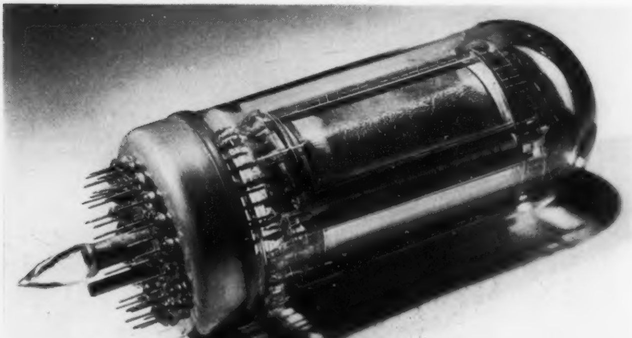

FIG. 2. An early experimental selectron.

Calculating Machine Solution of Quadratic and Cubic Equations by the Odd Number Method

1. Introduction. Eastlack[1] has published a method for the solution of quadratic equations by means of a calculating machine. His process is extended here to the solution of cubic equations. In the ordinary manual operation of calculating machines, the use of the method of solving cubic equations presented here will not be found to be as convenient as the use of certain other methods, such as that of Newton. The method is described here, however, in the belief that it may find application in large scale, automatic computing machines (such as the IBM Sequence Controlled Calculator or the ENIAC) where a large number of operations is not objectionable, provided that the operations are repetitive and sufficiently simple. We limit our discussion to real roots. Eastlack's method of solving quadratic equations is first reviewed so that the extension of the method to cubic equations may be clearer.

2. Extraction of square roots by the odd number method. The ordinary method for finding on the calculating machine the square root of a number of 3 or 4 digits of the form

$$(10x + y)^2 = 100x^2 + 20xy + y^2$$

consists in removing the $100x^2$ term by subtracting the x odd numbers, $1, 3, 5, \cdots, (2x - 1)$, from the first one or two digits of $(10x + y)^2$, and then removing the remainder of $(10x + y)^2$ by subtracting the y numbers, $(20x + 1), (20x + 3), \cdots, (20x + 2y - 1)$. This process of square root extraction may evidently be extended to the evaluation of roots having more than two digits.

The odd number method of square root extraction and a process of division may be combined expeditiously on a calculating machine to solve quadratic equations, of which we consider two types.

3. Positive root of $a = bx + x^2$ where $a, b > 0$. This quadratic equation has one positive root and one negative root. To illustrate the calculating machine solution of this equation we find the first six digits of the positive root of $3586 = 80x + x^2$. The successive steps of the solution are given in Table I. The number $a = 3586$ is entered on the machine as if for square root. Then from it are subtracted as in that process the numbers $1, 3, 5, \cdots$, each however combined with $b = 80$. The selection of columns is as follows: If the number b is placed under a so that their corresponding powers of 10 are aligned, the numbers $1, 3, 5, \cdots$, appear under the units column of a. If b is moved k places from this position, then the odd integers are moved $2k$ places in the same direction, so that they always appear under the right-hand digit of one of the pairs of digits in a. Of course k is taken as large algebraically as possible without the combined subtrahend exceeding a. Just as in square root, the subtrahend is increased by 2 before each subtraction, but when the next subtraction would produce a negative result the 2 is replaced by 1, 1 is added in the next column to the right, to whatever digit of b is already there, and the carriage is shifted. Just as in ordinary square root the successive digits of the result appear in the counting dials. As in

square root also when more than half the desired number of digits of the result have been obtained one may obviously cease to append any new sets of odd digits to the subtrahend and let the process degenerate into division.

TABLE I. Positive root of $3586 = 80x + x^2$

Keyboard (1)	Counting Dials (2)	Adding Dials (3)	Digit Added to Keyboard (4)	Accumulation of Column (4) (5)
080		3586	01	01
090	1	2686	02	03
110	2	1586	02	05
130	3	286	01	06
140	3	286	001	061
141	31	145	002	063
143	32	2	001	064
144	320	20000	00001	06401
14401	3201	5599	00001	06402
14402	32013	12784		
14402	320138	12624		

4. Smaller positive root of $a = bx - x^2$ where a, $b \geqq 0$ and $4a < b^2$. This quadratic equation has two positive roots. Quadratic equations having two negative roots may be changed to this form by changing the signs of the roots. The smaller root may be obtained by the process outlined in 3, with the exception that the odd numbers, 1, 3, 5, etc., are this time to be subtracted from the number b which is set on the keyboard. The subtraction of these odd numbers at no stage causes the number on the keyboard to become negative since $b > 2x$ if x is the smaller of the two positive roots.

5. Extraction of cube roots by the odd number method. Suppose that $10x + y$ is an integer of two digits, x being the first digit and y the second digit. Then

$$(10x + y)^3 = 1000x^3 + 300x^2y + 30xy^2 + y^3$$

may be regarded generally as an integer of six digits if in particular cases the first one or two of the six digits are zero. Now the portion of the first triad, or group of three digits arising from the term $1000x^3$ may be represented by the series

$$1000x^3 = 1000[1 + (1 + 6) + (1 + 6 + 2\cdot 6) + \cdots + (3x^2 - 3x + 1)],$$

having x terms. We may thus determine x by subtracting successively from the first group of 3 digits, 1, 7, 19, 37, $\cdots$, $3x^2 - (3x - 1)$, the number x of subtractions being recorded in the counting dials. The subtrahends themselves are built up by adding successively to the original 1 the numbers 6, 12, 18, $\cdots$, $6(x - 1)$, the number added being always 6 times the number currently in the counting dials. When $1000x^3$ is thus removed the remainder is $300x^2y + 30xy^2 + y^3$. For values of $y = 1, 2, \cdots$, this has the successive values $300x^2 + 30x + 1$, $600x^2 + 120x + 8$, $900x^2 + 270x + 27$. We may therefore find y by subtracting successively the initial value $300x^2 + 30x + 1$, and the consecutive differences, $300x^2 + 90x + 7$, $300x^2 + 150x + 19$, $\cdots$, these successive subtrahends being built up by adding to the original $300x^2 + 30x + 1$ on the keys the successive values $60x + 6$, $60x + 12$, $\cdots$, $60x + 6(y - 1)$, this number being 6 times that currently present on the counting dials. The initial value in this sequence of subtrahends is obtained from the final value of the earlier sequence. The latter is $1000(3x^2 - 3x + 1)$. To this we add $1000(3x - 1)$ and get $3000x^2$. Shifting the carriage one place to the left makes this effectively $300x^2$, and to this we add $30x + 1$ to get

the first subtrahend of the new sequence. This process of cube root extraction may evidently be extended to evaluate roots having more than two digits.

We apply this method of cube root extraction to the solution of three types of cubic equations in which the quadratic term has been removed.

6. Smaller positive root of $a = bx - x^3$ where $a, b > 0$ and $27a^2 < 4b^3$. This cubic equation has three real roots, two of which are positive and one negative. Similar cubic equations having two negative roots and one positive root may be transformed to this form by changing the signs of the roots. To illustrate the calculating machine solution of this equation we find the first six digits of the smaller positive root of $95242 = 4000x - x^3$. The successive steps of the solution are given in Table II. As in the quadratic case the constant term b is combined on the keys with the variable subtrahend which is to erase x^3. Since the latter is built up from its differences the term b is merely put into the first value and remains there. In this particular type of equation the cubic term has a minus sign. Hence the successive differences are subtracted from, instead of being added to the subtrahend. The rule for placing the columns is very similar to the quadratic case. If b has its columns aligned with the corresponding ones in a, then the numbers 1, 7, 19, $\cdots$ are in the units place. If b is moved k columns from that position then these numbers are moved $3k$ columns in the same direction. As in the quadratic case k is initially given the largest value that leaves the subtrahend less than a. When a digit in the result is obtained, the carriage is moved one place to the left. This moves b in effect one place to the right compared to the residue of a. The new sequence 1, 7, 19, $\cdots$ is then put in on the keys two places to the right, and hence three places to the right compared to a. The process of subtracting multiples of the partial root from the keyboard is most easily performed digit by digit, so that it is never necessary to subtract a number greater than 54. At the stage when 3201 appears on the counting dials, four of the required six digits of the root have been obtained, and two more digits may be obtained correctly by dividing the remainder 0.729601 on the carriage by the number 926.0797 appearing on the keyboard. Thus $x = 32.0107$ is the smaller positive root to six digits. The remaining roots are most conveniently obtained by eliminating this root and solving the resulting quadratic equation.

TABLE II. Smaller positive root of $95242 = 4000x - x^3$

Keyboard (1)	Counting Dials (2)	Adding Dials (3)	Digits Subtracted from Keyboard (4)
4000		95242	01
3900	1	56242	06 = 6 Col. (2)
3300	2	23242	12 = 6 Col. (2)
2100	3	2242	08 = 3 Col. (2) − 1
1300	3	2242	0091 = 30 Col. (2) + 1
1209	31	1033	0186 = 6 Col. (2)
1023	32	10	0095 = 3 Col. (2) − 1
0928	320	10000000	00009601 = 30 Col. (2) + 1
09270399	3201	729601	00009602 = 3 Col. (2) − 1
09260797	320107	81345	

Note that the subtraction of the terms of the odd number sequence at no stage causes the number on the keyboard to become negative since $b > 3x^2$ if x is the smaller of the two positive roots.

7. Real root of $a = bx + x^3$ where $a, b > 0$. This equation has one real root which is positive. Similar cubic equations having but one real root

which is negative may be transformed to this form by changing the signs of the roots. The root may be obtained by the process outlined in **6**, with the exception that the terms of the odd number sequence, 1, $(1 + 6)$, $(1 + 6 + 2\cdot 6)$, etc., are this time to be added to the number b which is set on the keyboard.

8. Real root of $a = x^3 - bx$ where $a, b \geqq 0$ and $27a^2 > 4b^3$. This cubic equation has one real root which is positive. Similar cubic equations having but one real root which is negative may be transformed to this form by changing the signs of the roots. To illustrate the calculating machine solution of this type of cubic we find the first six digits of the real root of $32541 = x^3 - 9x$. The successive steps of the solution are given in Table III. The number $a = 32541$ is registered on the adding dials. Set the number 100 on the keyboard, placing the unit hundreds digit, which is the first member of the odd number sequence, 1, $(1 + 6)$, $(1 + 6 + 2\cdot 6)$, etc., in the column beneath the digit 2 of 32541 on the carriage. The number $b = 9$ is subtracted from the keyboard, giving a new number 91 which is subtracted from the carriage. The process of adding terms of the odd number sequence to $-b$ on the keyboard, and subtracting the resulting numbers from the remainder on the carriage is now continued as indicated in Table III. The real root is $x = 32.0199$, the last two digits being obtained by the division process described in **6**.

TABLE III. Real root of $32541 = x^3 - 9x$

Keyboard (1)	Counting Dials (2)	Adding Dials (3)	Digits Added to Keyboard (4)
(− 0009)		32541	0100
0091	1	31631	0600 = 6 Col. (2)
0691	2	24721	1200 = 6 Col. (2)
1891	3	5811	0800 = 3 Col. (2) − 1
2691	3	5811	0091 = 30 Col. (2) + 1
2782	31	3029	0186 = 6 Col. (2)
2968	32	61	0095 = 3 Col. (2) − 1
3063	320	61	00009601 = 30 Col. (2) + 1
30639601	3201	30360399	00009602 = 3 Col. (2) − 1
30649203	32019	2776116	
30649203	320199	17688	

In the above illustration the number b was taken small enough relative to the number a so that the first term of the odd number sequence was greater than b, and hence could be combined with $-b$ on the keyboard. It is evident, however, that b may be greater than several terms of the odd number sequence even though $3x^2 > b$ where x is the real root. In this event it is necessary to subtract several terms of the odd number sequence from the carriage and independently add the number b to the carriage the same number of times, until a term of the odd number sequence is arrived at which is greater than b, and hence can be combined with $-b$ on the keyboard. The digits of the root will appear correctly on the counting dials, if zero is removed from the carriage every time that b is added to the carriage.

W. E. BLEICK

Naval Postgraduate School,
U. S. Naval Academy,
Annapolis, Md.

[1] ALLEN C. EASTLACK, [Methods of approximating roots of a quadratic equation, with a calculating machine], Amer. Inst. Actuaries, *Record*, v. 24, 1935, p. 21–23.

Film Slide Rule

Mechanical aids to computation are generally divided into two classes, namely, discrete devices and continuous devices. The former class includes the desk calculator and other machines based on a counting process. The latter includes such instruments as the ordinary slide rule and others that depend upon measurement of some continuous quantity like length. Discrete computing aids are characterized by accuracy that is theoretically unlimited, being bounded in practice only by bulk and cost, whereas continuous devices are limited by the inherent errors in the measurement of the basic physical quantity.

There appear to be very few aids to computation that combine the two principles. One combination instrument was devised by the writer while acting as Technical Aide to the National Defence Research Committee (NDRC). The objective was an instrument capable of handling functions, such as log x, x^2 etc., with errors roughly one tenth those of the ordinary 20″ slide rule, and with less eye strain and fatigue.

The resulting instrument is called the "Film Slide Rule," and, as its name implies, it uses movie film as the base for slide-rule scales. Each scale is printed on a separate film approximately 220 feet long and the films are wound on take-up reels.

If the films were laid side by side as in an ordinary slide rule, and a solution obtained by measuring off lengths of film, then the device would be an ordinary continuous computer. It would be subject to errors due to expansion of the film etc. Instead of making the length the basic quantity, therefore, we use the teeth on a sprocket that carries the film as basic. These are discrete units that can be counted, and not measured. The scale on the film acts in a dual capacity: primarily, it counts the sprocket teeth that pass under a fixed mark when the slide rule is set, and secondarily, it measures fractional parts of the distance between sprocket teeth. Finally, the scale defines a function of the basic variable.

Film Slide Rules have been made in various sizes, with three to ten films on each, and with appropriate mechanical connections between the sprockets to solve various problems. Scales have been made up to represent x^2, log x, log sin x, log cos x and log tan x, for use in solving triangulation problems in two and three dimensions. They have been found to save 80% to 90% of the time required for computation by the use of tables and desk calculators, when a large group of reasonably similar numerical values are given. An example of such a set of problems is found in the triangulation of, say, a hundred positions of an object moving in space.

The Film Slide Rule has the disadvantage, common to most continuous instruments, that it must be adapted to a special class of problems, and hence is of no value unless a large number of similar problems is to be solved.

G. R. Stibitz

393 S. Prospect St., Burlington, Vt.

Editorial Note: These models are in use at Fort Bliss, Texas. A description by Army Ground Forces Board, no. 4, Ft. Bliss, was issued May 9, 1947 by the Applied Physics Laboratory of The Johns Hopkins University. A report, *Stibitz Computing Machine, Model B, Designed and Built by Department of Physics, University of N. C., Chapel Hill, N. C.*, 1944, 12 leaves, gives details of the construction and operation. The description of last May, 10 leaves, includes excellent reproductions of photographs of the Ten Film Stibitz Calculator. But the present statement is the first one made for the general public.

On a Scarce Factor Table

The table in question is by J. Ph. KULIK and has the following title: *Divisores numerorum decies centena millia non excedentium. Accedunt tabulae auxiliares ad calculandos numeri cujuscunque divisores destinatae. Tafeln der einfachen Factoren aller Zahlen unter Einer Million nebst Hülfstafeln zur Bestimmung der Factoren jeder grösseren Zahl.* It was published at Graz (Austria), in 1825, and contains xxvi, +2+ 286 p. of size 13.4 × 20.7 cm. See *MTAC*, v. 2, p. 59f.

This table is not mentioned by GLAISHER in his list "On factor tables"[1]; nor by CUNNINGHAM in his paper "Factor tables. Errata"[2]; nor even by LEHMER "On the history of factor tables".[3] But the work is noted by HENDERSON in his bibliographic list,[4] is mentioned by DICKSON[5] and is listed by Kulik in "POGGENDORFF."[6]

This table appeared 14 years after the publication of the famous table of L. CHERNAC: *Cribrum arithmeticum*, giving the *complete* factorization of all the numbers not divisible by 2, 3 or 5 up to 1020000; and 8 years after the publication of *Table des diviseurs pour tous les nombres du premier million* of J. C. BURCKHARDT. Hence there was no need for a third table covering the same region of the natural numbers. In the Latin Preface K[7] writes that the table of B is folio (?) and that it must be consulted more than twice to find the complete factorization of a number having more than two prime factors. Therefore K apparently planned a different tabular arrangement so that only one or two consultations should be necessary to get the complete factorization of a number.

In his book K first gives a nine-page table of the complete factorization of all *composite* numbers not divisible by 2, 3, 5 or 11, up to 21500. Then follows a 257-page table of factors of the numbers not divisible by 2, 3, 5 or 11 from 21500 up to one million; the last two figures of the numbers are printed on one separate page of somewhat larger size than the other pages. If a number has more than two prime factors this is indicated by a point following the printed factor, and K states that he always selected that factor, the residual factor of which lies below 21500, the upper limit of the nine-page table, so that *only one* division would suffice to have all the prime factors. In the Introduction, §11, we read that the second table gives *only one* factor, the *smallest* one, but (in §13) that in some cases the number has factors too small to make the residual factor <21500 and that in *most of these cases* the table gives two factors. I wish to point out: (i). In many cases the residual factor of the given factor is not below 21500; (ii). In many cases the table gives not the smallest factor, but a larger one or two prime factors. (iii). There are cases in which it is necessary to consult the table three times. I shall later confirm these three assertions by examples. In §12 we read that the point, mentioned above, following the factor, is not always clearly printed but that in this case the factor is printed as much to the left as would be necessary for the point; e.g. the fifth column and sixth row of page 180 where the factor 43 is printed to the left and without the point.

The size of the book is unusually small. To save space K uses (but not

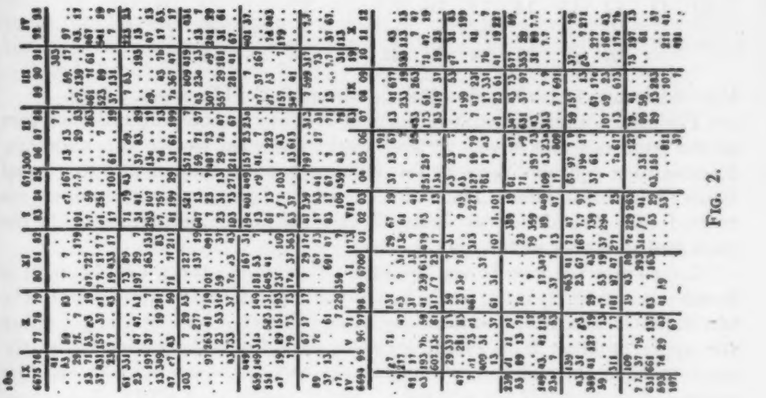

Fig. 2.

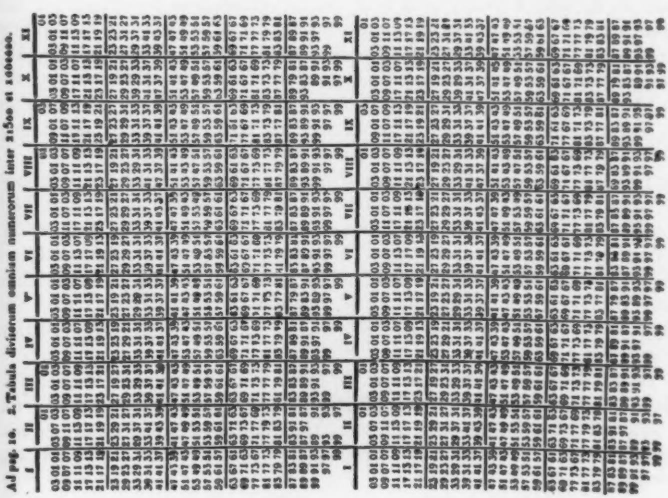

Fig. 1.

consistently) literal symbols:

	a	b	c	d	e	f	g	h	i	to denote the factors					
	13	17	19	23	29	31	37	41	43	and					
a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*k*	*m*	*n*	*o*	*p*	*q*	for
10	11	12	13	14	15	16	17	18	19	21	22	23	24	25	
		r	*s*	*t*	*u*	*v*	*w*	*y*	*z*	*A*	*B*	*C*	for		
		26	27	28	29	30	31	33	34	35	36	37			

E.g.: *b*3 means 113; *r*9 means 269; 7c means 7·19; 17e means 17·29.

The page is divided into two parts by a heavy horizontal line. Each part of the page contains seven vertical sections headed by Roman numerals. Each section contains three or less columns and is divided into five equal boxes by horizontal lines. The digits, except the last two, of the numbers are to be found just above two heavy horizontal lines, one at the top of the page and the other in the middle.

Let us now see how to use the table. Take for instance 670177; 6701 is found on page 180 (Fig. 2) at the top of the first column of section VII. The last digits 77 are to be found in the same column of section VII (of either the upper or lower half) of the separate page (Fig. 1). Having found the place of 77 in this column, we find in the corresponding place in the first column of section VII the symbol .., which means that 670177 is a prime.

Again let us take 670423; 6704 is at the top of the first column of section VIII (this VIII is omitted to save space). On the separate page in the first column of section VIII, we look for the last two digits 23. Then on page 180 in the corresponding place in the first column of section VIII we find the symbol 13a, that is to say 670423 has the factor 13·13. The residual factor is 3967. Entering the nine-page table we see that 3967 is omitted; hence it is a prime and we have the complete factorization 670423 = 13·13·3967.

The printed factor is not always the smallest one. Take for example 668423; 6684 is at the top of the second column of section I, on page 180. On the separate page the second column of section I look for the number 23. In the corresponding place on page 180, second column of section I, we find 41. indicating that 668423 has the factor 41 and that it is divisible by more than two prime factors. The residual factor is 16303 and the nine-page table gives 16303 = 7b·137. Hence the complete factorization 668423 = 41·7·17·137.

The residual factor is not always <21500. For the number 996659 the table gives 17., the residual factor is 58627. The second table gives the factor 23 not followed by a point, which means that 58627 = 23·2549 in primes. The same for 997441 for which the table gives 17., the residual factor being 58673 = 23·2551.

There are cases in which the table must be consulted three times. Thus for 986453 the table gives 13., the residual factor being 75881 above the upper limit of the nine-page table. The table gives for 75881 again 13., the residual factor is now 5837, for which the nine-page table gives 13·449. Again for 973271 the table gives 13., the residual factor being 74867, and for this number the table gives 13.; the residual factor is 5759 for which the nine-page table gives 13·443.

K gives no explanation as to the manner in which he obtained the entries. As we have pointed out, the main table does not always give the smallest prime factor, and it may give two prime factors. In view of the additional fact that some factors are followed by a point to indicate that the number has more than two prime factors, it seems clear that for the *calculation* of the entries it would have been necessary to *recalculate* the entire table of C. However we do not find anywhere in the book that K did this. Therefore it seems probable that he borrowed all his entries from C's table.

K gives a list of errors in C's table, containing all the errors detected 8 years earlier by B, but K added two new ones: 311909 *loco* 13·23·993 *lege* 13·23993 and 445193 *loco* 59639 *lege* 63599 (I saw in 2 copies of C's table 659 9, and in a third one 63599 which is correct). None of these errors is listed by D. H. Lehmer.[9]

K has not indicated any error in B's table.

There are additional tables as follows: a table of the squares up to 7500^2 in ten pages; a table of the linear forms of the divisors of $x^2 + ay^2$ up to $a = 106$ and of $x^2 - ay^2$ up to $a = 101$. K mentions (p. xxi) the table of LEGENDRE[8] and gives a list of errors in this table, containing a part of the list of errors published by D. N. Lehmer.[9] However the table of K contains numerous other errata. Further, the book contains a small table of the primes up to 3761, and a table of the powers of 2, 3, 5 up to 2^{71}, 3^{37}, 5^{37}.

In the Introduction K explains at length how the auxiliary tables may be used to factor a number beyond the range of the table, after having given rules on divisibility by 16, 9, 11, 101, 37, etc. He explains how to find quadratic forms for a given number and how to use them to get quadratic residues and linear forms of the divisors. For instance he factorizes $3^{24} - 1$ and identifies 10 091 401 as a prime.

K devotes only 30 lines (half a page) of the *Praefatio* to the description of the table. In the remaining $2\frac{1}{2}$ pages he explains in detail how he constructed another table giving a factor of every number not divisible by 2, 3 or 5 up to 30 030 000. As there are 80 numbers not divisible by 2, 3 or 5 among every 300 numbers $300v + 1, 2, 3, \ldots 300$, he made a sheet with 80 rows and 77 columns, headed 0, 3, 6, 9, . . . , denoting the hundreds. On this sheet he inserted the factors 7 and 11 in the proper places, and ordered copies printed (without the headings). These sheets are substantially the same as the sheet described by Glaisher.[1] K describes, with many technical details, the printing of more sheets until he possessed 1300 sheets upon which all the factors 7, 11, 13, 17, 19, 23 were *printed*. They covered the set of the numbers not divisible by 2, 3 or 5 up to 30 030 000. He tells us further that he inserted the factors 29, 31, . . . , 503 by the aid of what we now call "stencils" and the larger factors by the so-called "multiple method."

Comparing this description with that of the famous 100-million manuscript of K given by D. N. Lehmer[3] it seems that the 30 million table is *not* a part of the 100 million manuscript. For in the above description by K, literal symbols for numbers are not mentioned at all, while K did use them throughout in the *100 million* manuscript. Moreover D. N. Lehmer mentions that K used "stencils" as far as 997 in the *100 million* manuscript.

Hence as early as 1825 K was in possession of a manuscript factor table up to 30 million.

N. G. W. H. BEEGER

Nicolaas Witsenkade 10
Amsterdam, Holland

[1] JAMES GLAISHER, *Factor Table for the Fourth Million*. London, 1879; "Mode of construction of the table," p. 9–17, including a description of sieves; "On factor tables," p. 17–28.
[2] A. J. C. CUNNINGHAM, *Mess. Math.*, v. 34, 1904, p. 24–31.
[3] D. N. LEHMER, *Factor Table for the First Ten Millions*. Washington, 1909, p. V, X–XIV (See *MTAC*, v. 2, p. 139–140).
[4] J. HENDERSON, in *Factor Table* by J. PETERS, A. LODGE & E. J. TERNOUTH, & E. GIFFORD. London, 1935.
[5] L. E. DICKSON, *History of the Theory of Numbers*. Washington, v. 1, 1919, p. 351.
[6] *Biographisches-Literarisches Handwörterbuch z. Gesch. d. exacten Wissen.*, Leipzig, v. 1, 1863.
[7] To abbreviate, K = Kulik, B = Burckhardt, C = Chernac.
[8] A. M. LEGENDRE, *Essai sur la Théorie des Nombres*. Paris, 1798, T.III, $x^2 - ay^2$, up to $a = 79$, p. [477–483] and T.IV–V, $x^2 + ay^2$, up to $a = 105$, p. [484–489].
[9] D. N. LEHMER, Amer. Math. Soc., *Bull.*, v. 8, 1902, p. 401–402. See also D. H. LEHMER, *Guide to Tables in the Theory of Numbers*. Washington, 1941, p. 159–160.

EDITORIAL NOTES: Some parts of Dr. Beeger's paper are here somewhat amplified. The main points of Kulik's plan to avoid a large-sized book, such as Burckhardt's, may be outlined as follows:

(1) If the number N is prime, $N = p$, then the corresponding space (in the meeting of the horizontal and vertical lines) is marked with two dots $\cdot\cdot$.

(2) If the composite N has only two prime factors, $N = p_1p_2$, $p_1 < p_2$, then $p_1 < \sqrt{1000000} < 1000$, and has therefore at most 3 digits, for which there is sufficient space; so it is entered in full (see $988027 = 991 \cdot 997$).

(3) If N has three factors, $N = p_1p_2p_3$, then even the smallest p_1 and p_2, i.e., $p_1 = 7$, $p_2 = 7$, make $p_3 < 1000000/49$ or < 20408; therefore p_3 is within the range of T. 1 (goes to 21500). Hence p_1 and p_2 are entered completely, but when necessary for saving space, Roman letters, a to i, are used.

(3a) However sometimes only p_2 is entered (although not the smallest) when $N/p_2 < 21500$, and after p_2 a dot is inserted in order to indicate that N/p_2 is composite and the other factors should be looked up in T. 1.

(3b) Moreover, the smallest p which gives $1000000/p < 21500$ is $p = 47$. Hence when we have a factor $\geqslant 47$, we can enter that p, although it may not be the smallest, and write a dot after that p, to indicate that N/p is not prime, to be looked up in T. 1. But if all the p's < 47 we may have to write two factors, and in such cases, we utilize the Roman letters a...i ≡ 13...43.

(3c) Again if we have one factor of 3 digits to be followed by a dot and as there is no space for 4 marks, italic letters *a*...*C* are used.

(4) If N has 4 or more p's the procedure is the same as for 3 p's, except that in these cases the two factors entered will *always* be followed by a dot, since N/p_1p_2 is *always* composite.

Since in a publication of this kind errors may easily be made it should be noted that the four examples of failures in Kulik's system, according to Dr. Beeger, may be justified by the following corrigenda:

In 996659 and 997441, *for* 17., *read* 17d
In 986453 and 973271, *for* 13., *read* 13a.

In his account of the mss. of JOHN THOMSON (1782–1855), see *MTAC*, v. 1, p. 368, J. W. L. GLAISHER notes that "it is a coincidence" that H. G. Köhler's *Logarithmisch-trigonometrisches Handbuch*, Leipzig, 1848 (first ed. 1847, see RMT **430**) contained a 9-page factor table of numbers not divisible by 2, 3, 5, 11, up to 21,524, while Thomson gave a factor table "differing very little" from it, up to 21,460. Both of these tables are practically identical with Kulik's T. 1, up to 21,500, referred to above. Did Thomson copy from Kulik or Köhler? Did Köhler copy from Kulik? Or were all three tables independently original? Of course Lambert (1770) gave a table of the least factors of all numbers not divisible by 2, 3, 5, up to 102000 (see RMT **432**).

Even after extensive inquiries we have been unable to find in the United States any copy of Kulik's work here discussed, and the only copy in Great Britain seems to be the one in the Graves Library of the University of London. For editorial checking Dr. Beeger kindly loaned us his personal copy, of which a film reproduction was made for the Library of Brown University.

A Note on the Inversion of Power Series

1. Introduction.

At one time or another most applied mathematicians are faced with the problem of calculating the coefficients of the series

$$x = b_1 y + b_2 y^2 + \cdots + b_n y^n + \cdots$$

when given the coefficients of the series

$$y = a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots, \qquad a_1 \neq 0.$$

At such times there exists a choice between two long methods. The computer who is faced with this problem very often may derive explicit formulae for the desired coefficients and substitute directly. This method has the drawback usually encountered in substituting in formulae, namely that the computations are usually unsystematic and therefore become tedious and subject to many errors. Furthermore the formulae become extremely long and complicated for coefficients of appreciable order.[1]

The chief purpose of this paper lies in presenting a method which will be especially useful to the person who is unwilling to derive complicated formulae or undergo the ordeal of substituting in them. The method which will be presented will enable the computer to obtain n coefficients of the inverse power series using only one page of computations with approximately $\frac{1}{2}(n + 1)^2$ numbers. Besides being compact, this method has the advantage of being systematic. Furthermore similar methods can be easily obtained for most formal calculations with power series.

2. Multiplication of Power Series.

The fundamental part of the method of inversion is a simple device used to multiply power series. Because this method and its applications are not as widely known and appreciated as they should be, we shall indicate more properties of this method than is necessary for inversion.

If we are given

$$y = a_0 + a_1 x + \cdots + a_m x^m + \cdots$$
$$z = b_0 + b_1 x + \cdots + b_m x^m + \cdots$$

then

$$u = yz = c_0 + c_1 x + \cdots + c_m x^m + \cdots,$$

where

$$c_m = a_0 b_m + a_1 b_{m-1} + \cdots + a_m b_0 = \sum_{i=0}^{m} a_i b_{m-i}.$$

We write the coefficients b_i in the first column and leave the second column for the c_i. We also take a strip of paper with the coefficients a_i written from bottom to top. (See figure 1.)

We may calculate c_m by adjusting the strip so that a_0 is adjacent to b_m and then accumulating the products of all a's and b's which are adjacent. A special case of great importance is that where the a_i and b_i are coefficients of the same power series

$$y = a_{0,1} + a_{1,1} x + \cdots + a_{m,1} x^m + \cdots$$

Then the c_i are the coefficients of y^2. If we multiply the series with the

$\cdot$
$\cdot$
$\cdot$
a_m
a_{m-1}
$\cdot$
$\cdot$
$\cdot$
a_1
a_0

b	c	
b_0	c_0	
b_1	c_1	
$\cdot$	$\cdot$	
$\cdot$	$\cdot$	
$\cdot$	$\cdot$	
b_{m-1}	c_{m-1}	
b_m		

FIGURE 1.

coefficients c_i by the original series we get the coefficients of y^3. Thus if we denote the coefficients of y^n by means of

$$y^n = a_{0,n} + a_{1,n}x + \cdots + a_{m,n}x^m + \cdots$$

we obtain, by our method of multiplying power series, the coefficients $a_{m,n}$ on a single sheet of paper. (See figure 2.)

$\cdot$
$\cdot$
$\cdot$
$a_{m,1}$
$a_{m-1,1}$
$\cdot$
$\cdot$
$\cdot$
$a_{1,1}$
$a_{0,1}$

	y	y^2		y^n	
coef. of 1	$a_{0,1}$	$a_{0,2}$	$\cdots$	$a_{0,n}$	$\cdots$
coef. of x	$a_{1,1}$	$a_{1,2}$	$\cdots$	$a_{1,n}$	$\cdots$
$\cdot$	$\cdot$	$\cdot$		$\cdot$	
$\cdot$	$\cdot$	$\cdot$		$\cdot$	
$\cdot$	$\cdot$	$\cdot$		$\cdot$	
coef. of x^{m-1}	$a_{m-1,1}$	$a_{m-1,2}$	$\cdots$	$a_{m-1,n}$	$\cdots$
coef. of x^m	$a_{m,1}$	$a_{m,2}$	$\cdots$	$a_{m,n}$	$\cdots$
$\cdot$	$\cdot$	$\cdot$		$\cdot$	
$\cdot$	$\cdot$	$\cdot$		$\cdot$	
$\cdot$	$\cdot$	$\cdot$		$\cdot$	

FIGURE 2.

By use of the movable strip we may also calculate the quotient z of the power series u divided by that of y, assuming $a_0 \neq 0$;[2] it is obvious that $b_0 = c_0/a_0$. Furthermore if we are given $b_0, b_1, \cdots, b_{m-1}$ we can calculate b_m by

$$\sum_{i=1}^{m} a_i b_{m-i} = c_m, \qquad b_m = \frac{1}{a_0}\left[c_m - \sum_{i=1}^{m} a_i b_{m-i}\right]$$

where $\sum_{i=0}^{m} a_i b_{m-i}$ is obtained by means of the movable strip in an obvious manner.

Thus we can calculate the reciprocal $1/y$ and then by successive multiplications $1/y^n$; in fact we can calculate the power series y^n for any real n by making use of the binomial expansion,

$$y^n = [a_0 + a_1x + \cdots + a_mx^m + \cdots]^n$$

$$a_0^n[1 + v]^n = a_0^n\left[1 + nv + \cdots + \binom{n}{r}v^r + \cdots\right].$$

Now we construct a table as in figure 2 for the function v. Above the top row, insert the numbers $\binom{n}{r}$ corresponding to the r-th column. Then the coefficient of x^m in y^n is $a_0^n[1 + w_{mn}]$ where $w_{mn} = \sum \binom{n}{r} a_{m,r}$ is the accumulated product of the terms in the m-th row with the corresponding numbers $\binom{n}{r}$. Note that each sum is a finite sum because v has no constant term and therefore $a_{m,r} = 0$ for $r > m$.

The above process is a particular example of the evaluation of $u(x) = f[g(x)]$ where f and g are power series. In general, if the coefficients for g and f are $a_{m,1}$ and b_m respectively, we have merely to replace $\binom{n}{r}$ in the above by b_r and the coefficient of x^m in u is $b_0 + w_m$ where $w_m = \sum_{r=1}^{\infty} b_r a_{m,r}$.

The method of multiplying two power series in one variable by a strip of paper can be extended to the multiplication of power series in two or even more variables. Suppose

$$u = \sum a_{m,n}x^m y^n, \qquad v = \sum b_{m,n}x^m y^n, \qquad uv = w = \sum c_{m,n}x^m y^n,$$

where

$$c_{m,n} = \sum a_{i,j}b_{m-i,n-j}.$$

To calculate $c_{m,n}$ we must consider instead of two columns a_i, b_i, the rectangular arrays $a_{i,j}$, $b_{i,j}$. The extension is quite simple. We write the $a_{i,j}$ and $b_{i,j}$ on two cards as in figure 3.

$b_{0,0}$	$b_{0,1}$	$\cdots$	$b_{0,n}$	$\cdots$
$b_{1,0}$	$b_{1,1}$	$\cdots$	$b_{1,n}$	$\cdots$
.	.		.	
.	.		.	
.	.		.	
$b_{m-1,0}$	$b_{m-1,1}$	$\cdots$	$b_{m-1,n}$	$\cdots$
$b_{m,0}$	$b_{m,1}$	$\cdots$	$b_{m,n}$	$\cdots$
.	.		.	
.	.		.	
.	.		.	

	.	.		.
	.	.		.
	.	.		.
$\cdots$	$a_{m,n}$	$a_{m,n-1}$	$\cdots$	$a_{m,0}$
$\cdots$	$a_{m-1,n}$	$a_{m-1,n-1}$	$\cdots$	$a_{m-1,0}$
	.	.		.
	.	.		.
	.	.		.
$\cdots$	$a_{1,n}$	$a_{1,n-1}$	$\cdots$	$a_{1,0}$
$\cdots$	$a_{0,n}$	$a_{0,n-1}$	$\cdots$	$a_{0,0}$

FIGURE 3.

The crosses indicate the portions of the a-card which are cut out, and it is quite evident that to get $c_{m,n}$ we have merely to place the a-card on the

b-card so that $a_{0,0}$ corresponds to $b_{m,n}$ and to accumulate the products of corresponding terms.

It is noteworthy that in these formal operations with power series there is no need to restrict ourselves to series of positive powers only.

3. Inversion.

It is a simple step to compute the coefficients of the inverse series now. First we consider the case $y = a_{1,1}x + a_{2,1}x^2 + \cdots$, i.e., $a_{0,1} = 0$ and $a_{1,1} \neq 0$. Thus we can avoid dealing with all the terms above the diagonal for they will be zero. In fact our sheet would come out as in figure 4 after the addition of an extra column at the end and an extra row at the bottom of the sheet.

$$\begin{array}{|c|} \vdots \\ a_{m,1} \\ a_{m-1,1} \\ a_{m-2,1} \\ \vdots \\ a_{1,1} \\ 0 \\ \hline \end{array}$$

	y	y^2		y^{m-1}	y^m		d_i
coef. of 1	0	0	$\cdots$	0	0	$\cdots$	
coef. of x	$a_{1,1}$	0	$\cdots$	0	0	$\cdots$	d_1
coef. of x^2	$a_{2,1}$	$a_{2,2}$	$\cdots$	0	0	$\cdots$	d_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
coef. of x^{m-1}	$a_{m-1,1}$	$a_{m-1,2}$	$\cdots$	$a_{m-1,m-1}$	0	$\cdots$	d_{m-1}
coef. of x^m	$a_{m,1}$	$a_{m,2}$	$\cdots$	$a_{m,m-1}$	$a_{m,m}$	$\cdots$	
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
b_i	b_1	b_2		b_{m-1}	b_m		

FIGURE 4.

Now, since $y = a_{1,1}x + a_{2,1}x^2 + \cdots + a_{m,1}x^m + \cdots$

$$x = b_1y + b_2y^2 + \cdots + b_my^m \cdots$$

$$\begin{aligned} x = {} & b_1a_{1,1}x \\ & + b_1a_{2,1}x^2 + b_2a_{2,2}x^2 \\ & + b_1a_{3,1}x^3 + b_2a_{3,2}x^3 + b_3a_{3,3}x^3 \\ & \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ & + b_1a_{m,1}x^m + b_2a_{m,2}x^m + b_3a_{m,3}x^m + \cdots + b_{m-1}a_{m,m-1}x^m + b_ma_{m,m}x^m \cdots \\ & \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \end{aligned}$$

Equating coefficients we have

$$b_1a_{1,1} = 1, \qquad b_1 = 1/a_{1,1}, \qquad \text{since} \qquad a_{1,1} \neq 0,$$

$$b_1a_{2,1} + b_2a_{2,2} = 0, \qquad b_2 = -b_1a_{2,1}/a_{2,2}, \qquad a_{m,m} = (a_{1,1})^m \neq 0, \quad \text{etc.}$$

Assuming we have $b_1, b_2, \cdots, b_{m-1}$, we use the m-th equation to get b_m

$$b_1a_{m,1} + b_2a_{m,2} + \cdots + b_ma_{m,m} = 0, \qquad b_m = -d_m/a_{m,m},$$

where $d_m = b_1a_{m,1} + b_2a_{m,2} + \cdots + b_{m-1}a_{m,m-1}$ is obtained by an accumulation of the products of the $(m - 1)$ b's, which we know, by the corresponding a's in the row of coefficients of x^m.

Thus we can calculate as many of the coefficients of the inverse series as we wish by this method, being careful only to take a sheet of paper which is large enough, i.e., having $(r + 1)$ rows and $(r + 1)$ columns for r coefficients. To recapitulate, this method permits us to calculate the coefficients of the inverse power series systematically and on one page. Furthermore in the calculations it requires only accumulations of products with the exception of r divisions.

Now consider the inversion problem where the coefficient of x is zero. If $z = d_nx^n + d_{n+1}x^{n+1} + \cdots$, we may, by the method described in 2, obtain

$$z^{1/n} = a_{1,1}x + a_{2,1}x^2 + \cdots,$$

where $a_{1,1} = d_n^{1/n} \neq 0$. Then we may obtain x as a power series in $z^{1/n}$.

H. CHERNOFF

Brown University

[1] FRANZ KAMBER, "Formules exprimant les valeurs des coefficients des séries de puissances inverses," *Acta Math.*, v. 78, 1946, p. 193–204.

[2] The case $a = 0$ is easily handled by factoring out x^n where a_n is the first non-zero coefficient.

EDITORIAL NOTE: A "movable strip" is extensively used by actuaries in their insurance and annuity calculations, in connection with their "commutation" columns. In actuarial literature there are frequent references to this "movable strip"; e.g., GEORGE KING, *Institute of Actuaries' Text Book*, part II, second ed., 1902, p. 392–393, 402.

RECENT MATHEMATICAL TABLES

For other RMT see ACM: Bibliography (Stibitz, NDRC, Zuse); OAC: Bibliography; N**75** (Horton) and **79** (Katz); QR**30**.

425[A].—A. ADRIAN, *Barème Forestier. Cubage des Bois abattus des Bois en grume d'après la Circonférence et le Diamètre et des Bois Équarris. Débit et Équarrissage des Bois. Cubage et Estimation des Bois sur Pied. Conversion du Volume réel.* Paris, Éditions Berger-Levrault, 54th thousand, 1944. iv, 214 p. 11.3 × 17.5 cm.

T. 1, p. 5–97 gives the volume in cubic meters, to 3D, of round wood of circumference $c = 25(1)300$ centimeters and length $l = .25(.25)16$ meters.

T. 2, p. 99–131, gives similar results for diameter $d = 5(1)100$ centimeters.

T. 3, p. 132–179, is for volumes of squared wood, $l = .25, .33, .5, .66, .75, 1(1)20$ meters, and cross sections $5 \times 5(1)11$ up to $50 \times 50(1)55$, 100 centimeters.

T. 4, p. 180–185, by three different methods of "squaring" round wood, from $c = 32$, $d = 10$, $(7 \times 7, 8 \times 8, 8 \times 9)$ to $c = 300$, $d = 95\frac{1}{2}(67 \times 68, 75 \times 75, 90 \times 91)$.

Miscellaneous small tables p. 190–212.

426[A].—R. C. MORRIS, "Table of multiples of the square root of three," *Electrical World*, v. 125, June 8, 1946, p. 108–109. 21.6 × 28.8 cm.

This is a table of $N\sqrt{3}$, where $\sqrt{3}$ is taken as 1.73205; $N = [1(.01)9.99; 4D]$. In MARCEL BOLL, *Tables Numériques Universelles*, Paris, 1947, p. 184–185, are 6D tables of $Nx^{\frac{1}{2}}/D$ for $x = 2, 3, 5$, N or $D = 1(1)10$.

427. [A, B, C, D, E, M].—E. S. Allen, *Six-Place Tables. A Selection of Tables of Squares, Cubes, Square Roots, Cube Roots, Fifth Roots and Powers, Circumferences and Areas of Circles, Common Logarithms of Numbers and of the Trigonometric Functions, The Natural Trigonometric Functions, Natural Logarithms, Exponential and Hyperbolic Functions, and Integrals. With Explanatory Notes.* Seventh ed., New York, McGraw-Hill, 1947. xxiii, 232 p. 10.7 × 17.7 cm. $1.75. Compare RMT **49**, **184**.

The seventh edition is 51 pages larger than the sixth, which was reviewed, *MTAC*, v. 1, p. 348. This is due to the expansions of T. X, Natural Logarithms (by 17 p.), and T. XI, Exponential and Hyperbolic Functions (by 34 p.), from 4D to 6D, with finer arguments. T. XIV, Mathematical Constants, has been slightly extended.

428[A–F, H, L–N].—Marcel Boll, *Tables Numériques Universelles des Laboratoires et Bureaux d'Études.* Paris, Dunod, 1947, iv, 882 p. 18.5 × 27.1. Bound in boards 3200 francs (about $27).

This well-printed, well-arranged, and excellently indexed volume contains more than 200 tables grouped under the following six general headings. A, "Arithmétique" and algebra, p. 7–218, where the tables are denoted by the numbers An, $n = 1(1)40$; **T**, Trigonometry, p. 219–386, $n = 1(1)38$; **E**, Exponentials, p. 387–536, $n = 1(1)44$; **P**, Probabilities, p. 537–686, $n = 1(1)40$; **C**, Complex numbers, p. 687–746, $n = 1(1)19$; **U**, Units, constants, p. 747–854, $n = 1(1)36$. Graphs of the functions discussed are numerous; there are no less than 122. The page numbers are in large black-face type at the bottom of each page and above every page of every table are page references to text or graphs or reliefs dealing with the function tabulated. All the printed numerals in the tables are of uniform block, black-face type. Black-face arrows at the bottom of each right-hand page indicate if the table is continued on the next page.

On p. 6, there is a general Bibliography with 39 titles. To the uninitiated a number of these titles might be thought to refer to comparatively recent works. For example, "Edouard Barbette, Sommes des $p^{\text{ièmes}}$ puissances (Paris, 1942)." But this work appeared originally in 1910, and the fuller title is *Les sommes de $p^{\text{ièmes}}$ Puissances Distinctes égales à une $p^{\text{ième}}$ Puissance: suivie d'une Table des 5000 premiers Nombres Triangulaires.* Liége and Paris. Or consider also the entry "J. Claudel, Tables des Carrés, Cubes, Longueurs de Circonférences, . . . (Paris, 1939)"; the original of this work appeared 89 years earlier, with somewhat different title. Claudel died in 1880. Four of the titles without dates (the only ones) are all German works, by Crelle, Peters, Landolt-Börnstein, and Schrön. The title of the Crelle work is given as "Produits des nombres de 1 à 1.000 par les nombres de 1 à 1.000 (Berlin)." Of course no book of Crelle with such a title was ever published; the possibilities of Crelle's *Rechentafeln* or *Calculating Tables* are thus set forth. So also for Peters' *Neue Rechentafeln für Multiplikation und Division* . . . (1909) the title of which is listed as "Produits des nombres de 1 à 10.000 par les nombres de 1 à 100 (Berlin)."

If all previously published tables in the volume were adapted from the bibliographic sources listed, it is obvious that many tables are new; and indeed the author states that more than a third of the pages in the volume are filled with such previously unpublished material. As the general title-survey given above suggests, the tables are mainly of an elementary nature—the more advanced ones being of two complete elliptic integrals, of Fresnel integrals, of sine, cosine, and exponential integrals, of Legendre polynomials, and of Bessel functions. But the elementary tables are often given because they are useful in various non-elementary applied fields, for example: Equation of Van der Waals (p. 204–206); corrections of relativity (p. 207–209); equation of paramagnetism, Langevin, 1905 (p. 472) greatly inferior to the table in Emde, *Tables of Elem. Functions*, 1940, p. 123; functions of Planck, 1901 (p. 493–502) supplementing the table in Emde, *Tables of Elem. Functions*, 1940, p. 117–119; curves of Einstein and Debye (p. 503–512); curves of Gauss and Galton (p.

580–632); curves of Poisson (p. 671–686). There is emphasis on tables useful for the physicist and chemist.

Some random indications of tables included in the volume are as follows: (A1), circumferences of circles of diameter $n = [1(1)1000; 6S]$; (A2), Surfaces and volumes of spheres of diameter $n = [1(1)100; 7S]$; (A11), triangular coordinates, triangles of J. W. GIBBS (1876), and BAKHUIS ROOZEBOOM (1894); (A19), $(a/b)^{\frac{1}{2}}$ and $(b/a)^{\frac{1}{2}}$, $a = 1(1)100$, $b = [2(2)30(5)100; 4D]$; (A20), $(a - 1/b)^2$, $a = .1(.1)10$, $b = .1(.02).3(.05)1$; (A25), $\sqrt{ab}$; (A26), $M = 2ab/(a + b)$; (A33), molecular refraction, $(x^2 - 1)/(x^2 + 2)$; (A34), molecular polarization, $(x - 1)/(x + 2)$; (A40), simple approximations of some incommensurable numbers ($\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, e, $\log e$, π, $\sqrt{\pi}$, $\sqrt{2\pi}$, π^2) with indications of the errors of the approximations; (T2), remarkable expressions (about 40) expressed as functions of π; (T16), $\sin^2 A$, $\cos^2 A$, $\cos^3 A$, $\cos^4 A$, $\cos^{\frac{1}{2}} A$, $\cos A \cos^{\frac{1}{2}} A$, $(1 - \tan A)/(1 + \tan A)$, $\sin A \cos A$, $\sin A \cos^2 A$, $\sin A \cos^3 A$, $\sin A \cos^{\frac{1}{2}} A$, $\sin A \cos^{-\frac{1}{2}} A$, somewhat like tables in the other French volume of over 800 pages of formulae and tables by L. POTIN, Paris, 1925, p. 407–417; (E 12), $u = x^x$, $v = x^{-x}$, u', v', $U = \int_0^X x^x dx$, $V = \int_0^X x^{-x} dx$; (E 20), $\log \log x$; (E 26), $x + \frac{1}{2}\sinh 2x$, $\sinh x/x$, $\tanh(\frac{1}{2}x)/\frac{1}{2}x$; (E 30), $\sinh x \cos x$, $\cosh x \sin x$, $\cosh x \cos x$, $\sinh x \sin x$; (E 42), $y = e^{-2\pi zx}\sinh[2\pi(z^2 - 1)^{\frac{1}{2}}x + \psi]/\sinh \psi$, where $\sinh \psi = (z^2 - 1)^{\frac{1}{2}}$; (P 31c), $(1 - p^\alpha)/[(1 - p)p^\alpha]$; (C 7–8), roots of equations of the third degree as in JAHNKE & EMDE, and also by another method; (U 12), $y = (e^{2x} - 1)^{\pm\frac{1}{2}}$; (U 22), $h = v^2/2g$, h in cm., v in cm/s, $g = 980.9$ cm./sec^2.

We have now given some suggestions as to the wealth of material in this volume, which, if accurate, might often be useful in very varied fields of work. The alphabetical index (p. 859–868) leads quickly to material in the volume which may be sought. There is also a list of the graphs and reliefs (p. 869–871) and a list of the tables in order (p. 873–882). The pages (856–857) headed "Interpolation précise des tables" do not make a favorable impression. Wholly random checking of a few of the tables (see below) shows that they are highly unreliable, displaying not only defective proof reading, but also carelessness and inadequate checking of basic calculations. Hence the reliability of no table in the volume should be assumed without careful checking. It looks as if Hayashi's throne has been lost to a Frenchman.

Marcel Boll (1886–) is also the author of: (i) *La Chance et les Jeux de Hasard . . .*, Paris, 1936, 386 p. (ii) *Le Mystère des Nombres et des Formes . . .*, Paris 1941; fourth ed., 1946, 330 p. (iii) *Les Étapes des Mathématiques*, Paris, third ed., 1944, 128 p.

R. C. A.

P. 134, $x = 7$, $p = 56$, *not* 46; $x = 8$, $p = 40$, *not* 38; $x = 14$, $y = 48$, *not* 38, and $p = 112$, *not* 102; $x = 15$, $z = 113$, *not* 115, $p = 240$, *not* 142; $x = 22$, $p = 264$, *not* 164; $x = 23$, $p = 552$, *not* 352; $x = 26$, $p = 364$, *not* 398.

P. 226, T. 2, l. 10, col. 2, *for* 0.063602, *read* 0.063662.

P. 234–239, T. 8, 6-place table of the six trigonometric functions, 0(1°)360°: In csc 5°, sec 85°, sec 95°, csc 175°, csc 185°, sec 265°, sec 275°, csc 355°, *for* end-figures 513, *read* 713; in sin 21° and 7 other angles, *for* end-figure 3, *read* 8; in csc 29° and 7 other angles, *for* end-figures 53, *read* 65; in sec 65° and sec 245°, *for* end-figures 62, *read* 02.

P. 310, 18°, $\sin^2 \alpha$, *for* 0.095496, *read* 0.095492; 19°, $\sin^2 \alpha$, *for* 0.106092, *read* 0.105995; 21°, $\sin^2 \alpha$, *for* 0.128425, *read* 0.128428.

P. 312, 28°, $\cos^2 \alpha$, *for* 0.779614, *read* 0.779596; 39°, $\cos^2 \alpha$, *for* 0.603953, *read* 0.603956; 41°, $\cos^2 \alpha$, *for* 0.569584, *read* 0.569587; 41° 30′, $\sin^2 \alpha$, *for* 0.439043, *read* 0.439065; $\cos^2 \alpha$, *for* 0.560985, *read* 0.560935; 48°30′, $\sin^2 \alpha$, *for* 0.560985, *read* 0.560935; $\cos^2 \alpha$, *for* 0.439043, *read* 0.439065; 49°, $\sin^2 \alpha$, *for* 0.569584, *read* 0.569587; 51°, $\sin^2 \alpha$, *for* 0.603953, *read* 0.603956; 62°, $\sin^2 \alpha$, *for* 0.779614, *read* 0.779596; 63°30′, $\cos^2 \alpha$, *for* 0.199090, *read* 0.199092; 67°30′, $\cos^2 \alpha$, *for* 0.146417, *read* 0.146447; 69°, $\cos^2 \alpha$, *for* 0.128425, *read* 0.128428; 71°, $\cos^2 \alpha$, *for* 0.106082, *read* 0.105995; 72°, $\cos^2 \alpha$, *for* 0.095496, *read* 0.095492; 87°, $\cos^2 \alpha$, *for* 0.002727, *read* 0.002739.

P. 318, 40°, col. 1, *for* 0.58740, *read* 0.58682; the same error in a whole line of multiples. P. 319, 50°, col. 1, *for* 0.41260, *read* 0.41318; the same error in a whole line of multiples. P. 320, 29°, col. 1, *for* 0.84895, *read* 0.84805, the same for p. 321, 61°, col. 1.

P. 400, $N = 36$, $n = 1$, *for* 606, *read* 666; $N = 8$, $n = 2$, *for* 240, *read* 204.

P. 634, in the first 50 entries of the 7D table of $H(x)$, attributed to DEMORGAN 1845, there are 36 end-figure errors, 25 unit errors, 10 2-unit errors, and 1 90-unit error ($x = .08$). All of these errors except the last one (where Boll substituted 871 for DeMorgan's correct 781) are also in DeMorgan's *An Essay on Probabilities, Cabinet Cyclopædia*, 1838, p. xxxiv.

P. 786, $t = -197$, *for* 0.27868, *read* 0.27878; $t = -196$, *for* 0.28248, *read* 0.28245.

S. A. J.

429[B, P].—L. VUAGNAT, "Courbes de raccordement," *Bull. Technique de la Suisse romande*, v. 73, 10 May, 1947, p. 127–128. 23 × 31.4 cm.

In this paper on railway transition curves, p. 128, there is a table of $C_2 = n^4(2n^2 - 6n + 5)$ for $n = [.08(.001).999; 6D]$, Δ. There is also a brief table of $C_1 = n^3(6n^2 - 15n + 10)$, $n = [.05(.05)1; 3D]$. The values of the general formulae for C_1 and C_2 are each given incorrectly in this paper.

430[C, D].—CARL CHRISTIAN BRUHNS (1830–1881), **a.** *Nuovo Manuale Logaritmico-trigonometrico con sette decimali.* Twenty-second stereotyped ed., Novi Ligure, Società editrice Novese, [1941], xxiv, 610 p. Preface to the Italian ed. by TITO FRANZINI, 15.7 × 24.3 cm. The Library of Congress spells the first name Karl.

b. *A New Manual of Logarithms to Seven Places of Decimals.* Revised ed. Chicago, Ill., Charles T. Powner Co., P.O. Box 796, 1942, xxiv, 610, 3 p. 15.6 × 24 cm. $2.00.

First of all it is to be noted that one of the bases of Bruhns' work was *Logarithmisch-trigonometrisches Handbuch* . . . Leipzig, Tauchnitz, 1847, xxxvi, 388 p. 15.8 × 24 cm. by HEINRICH GOTTLIEB KÖHLER (1779–1849). The second rev. stereotyped ed. was in 1848, the third in 1856, the fourth in 1855, the fifth in 1857, the eighth in 1862, the thirteenth in 1876, the fourteenth in 1880, the fifteenth in 1886, and the sixteenth in 1898. There were also Italian editions in which the author's name appears as E. T. Köhler since Heinrich Gottlieb = Enrico Teofilo.

The first editions both in German and in English of the work by Bruhns appeared in 1870: *Neues logarithmisch-trigonometrisches Handbuch auf sieben Decimalen.* Leipzig, xxiv, 610 p. Some extracts from the preface (written Aug. 1869) are as follows: "Köhler's Handbook of Logarithms, which has hitherto been published by Tauchnitz, and which will still be published by them, has always found a very favourable reception from the public both on account of its arrangement and its exactness. However Bremiker's edition of Vega's seven-figure logarithms [1852] extended and improved, well known and frequently used, and Schrön's logarithmic tables [1860], are preferable for many elaborate astronomical calculations. Bremiker gives in the trigonometrical tables the logarithms of the Sine and Tangent for the first 5 degrees to every second and the logarithms of the Sine, Tangent, Cotangent, Cosine from 0 to 45° (and therewith it is self-evident of the whole quadrant) for every 10 seconds; whilst Schrön has added to the last an extensive Interpolation Table."

"The publishers did not wish to be behindhand with their Handbook of logarithms and when they became aware that I was willing to undertake the necessary labour of preparing one—they determined to preserve Köhler's in its present form . . . and desired me to prepare an entirely new Manual. . . . The logarithms of numbers from 1 to 108000 as they are in Köhler have been reduced to the extent of from 1 to 100000. . . . The logarithms of the first 6 degrees of the trigonometrical functions, Sine, Cosine, Tangent and Cotangent have been given to every second [in Köhler only to every 10″], with the addition of the differences and where the space would allow of it, of the proportional parts. . . . The remaining 39 degrees of the trigonometrical functions are given to every 10 seconds, whilst in Köhler from the 9th degree they are only given to every minute. . . . We have omitted the Addition and Subtraction logarithms (Gauss's) . . . and we have also omitted the

goniometrical and trigonometrical formulae contained in Köhler, as well as the other tables which though useful are not so frequently wanted; so that the present work with the exception of some few small additional Tables consists merely of the logarithms of numbers and of the trigonometrical logarithms."

The second English, German (Henderson has 1880), and Italian editions were in 1881, and the second French edition in 1880; apparently all of these language editions were, after 1881, given simultaneously successive edition numbers. The third edition was in 1889, the fourth in 1894, the seventh in 1906, the eighth in 1909, and the ninth in 1910. An eleventh English edition in 1913 had on the title-page both the names of Tauchnitz, Leipzig, and Lemcke & Büchner, New York. In this edition we find the following: "For the first edition, the publisher fixed one Friedrichsd'or as a prize for finding a typographical error, and since 1870 there have only been found 6 errata. In this edition these misprints are corrected." These were already corrected in the second edition.

The eleventh French edition, 1913, bore the title: *Nouveau Manuel de Logarithmes à Sept Décimales pour les Nombres et les Fonctions Trigonométriques*. A twelfth English edition was published by Van Nostrand, New York, in 1919, a thirteenth in 1922, and a sixteenth English edition by Regan Publ. Corp., Chicago in 1929. What edition Powner copied has not been determined. He issued copies also dated 1936, 1939, and 1941. The three pages added at the end, in the 1942 Powner print, white on black, gave "Table for converting minutes and seconds into decimals of a degree," "Decimal equivalents of an inch and corresponding logarithms"; "Trigonometric lines," and "Trigonometric functions of all angles."

Reporting on Bruhns' work Glaisher wrote, "On the whole, this is one of the most convenient and complete (considering the number of proportional-part tables) logarithmic tables for the general computer that we have met with; the figures have heads and tails; and the pages are light and clear."

R. C. A.

431[C, D].—MARIO O. GONZÁLEZ, *Nuevas Tablas de Logaritmos y de Funciones Naturales*, Havana, Cuba, Editorial Selecta, O'Reilly 357, 1945. xxxii, 98 p. Bound in Boards $1.00.

This volume contains the following five tables, mostly to five places: **T. 1**, log N, $N = 1000(1)10,009$, with P.P.; **T. 2**, logarithms of trigonometric functions sine, tangent, cotangent, cosine with Δ and P.P.:, at interval 1′; **T. 3**, Log S, log T at interval 1′ for 0 to 3°; **T. 4**, Natural trigonometric functions, at interval 1′, 5D for all functions 0–45°, except sin, cos, tan 6D for 0–2°, and cot 4D for 0–6°; **T. 5**, (a) n^2, (b) $n^{\frac{1}{2}}$, (c) $(10n)^{\frac{1}{2}}$, (d) n^3, (e) $n^{\frac{1}{3}}$, (f) $(10n)^{\frac{1}{3}}$, (g) $(100n)^{\frac{1}{3}}$, for $n = 1(1)100$; (b) and (e) are to 6D, (c) and (f) to 6 or 7S.

432[D, F, L, R].—J. H. LAMBERT, *Mathematische Werke, I. Band: Arithmetik, Algebra, und Analysis.* Ed. by ANDREAS SPEISER. Zürich, Füssli, 1946, Portrait + xxxii, 358 p. 15.5 × 22.9 cm.

JOHANN HEINRICH LAMBERT (1728–1777), German physicist, mathematician and astronomer, was born in Mülhausen, Alsace, and largely self-taught. During 1748–1756 he was tutor to children of Count de Salis at Coire, Switzerland; in 1753 he was elected a member of the Swiss Society of Basle and later contributed several memoirs to *Acta Helvetica*. During 1756–1758 he traveled, sojourning at universities of Germany, Holland, France and Italy. In 1761–1763 he spent some time again at Coire and Zürich but from 1764 to the end of his life he was almost wholly at Berlin where he received many favors from Frederick the Great and was in 1765 elected a fellow of the Royal Academy of Sciences. At this time Euler and Lagrange were also active in Berlin. The table-maker J. C. SCHULZE (1749–1790) was a pupil of Lambert.

In 1761 Lambert proved that π was irrational, although as early as 1689 J. C. STURM, in his *Mathesis Enucleata*, stated (p. 181) "Area circuli est quadrato diametri incommensurabilis." He introduced hyperbolic functions into trigonometry and published the first table of such functions (1770). He made geometrical discoveries of value, and theorems

concerning conic sections bear his name (see C. TAYLOR, *Introd. to the Ancient and Modern Geometry of Conics*, Cambridge, 1881). Astronomy was enriched by his investigations.

From what has been indicated above it is not inappropriate that a Swiss publisher and a Swiss scholar should collaborate in bringing out first volumes of Lambert's publications in the fields of "Arithmetik," Algebra and Analysis. Later volumes are planned to care for his writings in Applied Mathematics, Astronomy and Physics, and in Philosophy and Logic. We shall now refer only to the tabular material in four places of the present volume. All of this material appeared originally in Lambert's *Beyträge zum Gebrauche der Mathematik und deren Anwendung*. Berlin, 3 v., 1765–1772. The first three portions were in the second v., 1770, and the last in the third, 1772.

I, *Vorschlag, die Theiler der Zahlen in Tabellen zu bringen* p. 117–132. In 1767 HENDRIK ANJEMA's *Table of Divisors of all the Natural Numbers from 1 to 10000* (vi, 302 p.) was published. This table gave every divisor for each number, even 1 and the number itself. Since Lambert felt that a very small table would readily produce all the essential facts of this volume, he here gives his substitute (omitting all numbers divisible by 2, 3, 5) on a single folded sheet 34 × 51 cm. in size for the printed part. This is arranged in 9 facsimile pages (124–132) in the Speiser edition. Speiser also notes the 29 errors in Lambert's table which J. WOLFRAM sent to Lambert in a letter of 3 August 1772, printed in *Deutscher gelehrter Briefwechsel Joh. Heinr. Lamberts*, ed. by JOHANN (III) BERNOULLI, v. 4, 1784, p. 448. In this table all the prime factors are sometimes given for the non-prime numbers. In the year 1770 Lambert published also *Zusätze zu den logarithmischen und trigonometrischen Tabellen.* Table I in this collection gives the smallest factor of all numbers less than 102000 and not divisible by 2, 3, or 5. Concerning this volume LAGRANGE wrote as follows to D'Alembert on 4 April 1771: "J'y joindrai aussi un Ouvrage de M. Lambert qui a paru l'année passée, et qui n'est qu'une collection de différentes Tables numériques qui peuvent être très-utiles dans plusieurs occasions; c'est moi qui lui en ai donné l'idée et qui l'ai excité à l'exécuter." [LAGRANGE, *Oeuvres*, v. 13, 1882, p. 195. To the word "passée" the editor, J. A. SERRET, has added the following absurd footnote: "*Observations trigonométriques.* Lu à l'Académie de Berlin en 1768 et imprimé (p. 327–356) dans le volume portant la date de cette année, qui ne parut qu'en 1770."]

In ANTON FELKEL's Latin edition of Lambert's *Zusätze* (Lisbon 1798), T. I. has been elaborated by the indication of many extra factors, partly through the use of 36 letters standing for prime numbers up to 173, a scheme similar to that used by Felkel in his factor table of 1776–1777 (see *Scripta Mathematica*, v. 4, 1936, p. 336–337).

II, *Algebraische Formeln für die Sinus von drey zu drey Graden*, p. 189–193. On p. 190–191 is a table of sin $n3°$, for $n = 1(1)29$, expressed as functions involving only the square roots of numbers. Speiser corrects one error in the last entry. The table is reprinted in *Zusätze* p. 137–138, and in the 1798 edition, p. 125–126.

III, *Vorläufige Kenntnisse für die, so die Quadratur und Rectification des Circuls suchen.* The table on p. 204–205 contains 27 entries of various ratios, continually closer approximations to the ratio of the diameter to the circumference of a circle. These values (1:3, 7:22, 106:333, 113:355, $\cdots$) are the successive convergents of the regular continued fraction for π^{-1} which Lambert gives. In the second last term of this fraction Lambert made a numerical slip (pointed out by Wolfram in the letter of 3 August 1772, referred to above) which vitiated the accuracy of his last two ratios. As Speiser notes, these should have been

4448 54677 02853:13975 52185 26789
13630 81215 70117:42822 45933 49304.

The last ratio gives π^{-1} correct to 29 decimal places. In his edition of *Archimedes, Huygens, Lambert, Legendre. Vier Abhandlungen über die Kreismessung*, Leipzig, 1892, Rudio failed to observe the errors noted above and hence made a misleading statement. The first 16 of the entries of Lambert's table are in his *Zusätze*, p. 145, and in the 1798 edition, p. 133.

Both Speiser and Rudio are guilty of a curious oversight, in failing to refer to the table which JOHN WALLIS gave, 85 years before Lambert, in his *A Treatise of Algebra* . . ., London, 1685, p. 51–55; (see D. H. LEHMER, "Euclid's alogrithm for large numbers,"

Amer. Math. Mo., v. 45, 1938, p. 227–233). Not only do we here find the whole of the table of Lambert, and without any error, but 7 terms more, of which six are correct; the last correct ratio (the 33rd) is

842 46858 74265 13207:2646 69312 51393 04345;

from this, π may be determined correctly to 38D. It may be determined to 95D from the 91st ratio given by D. H. L. in the article indicated above.

IV, *Rectification elliptischer Bogen durch unendliche Reihen*, p. 312–325. The treatment of this problem is illustrated by a problem in geodesy. On p. 324 is a facsimile reproduction of a table giving for each degree of latitude the polar distance with its first difference which is approximately the length of one degree of longitude at that latitude. The earth is assumed to be a spheroid whose meridian section is an ellipse whose axes are in the ratio 230/229, an assumption attributed to NEWTON. Distances are given to the nearest klafter, an archaic French unit of length equal to 0.3875 rod. Unfortunately the whole table is slightly wrong, each polar distance being too large by about .047 percent, as noted by Speiser, on account of a small error in the formula on which the table is based.

R. C. A. & D. H. L.

433[E, M].—L. LANDWEBER & M. H. PROTTER, "The shape and tension of a light flexible cable in a uniform current," *Jn. Appl. Mechanics*, v. 14, June, 1947, p. A123–A124.

$$\tau = e^{(1/45)\cot\phi} = e^{u/45},\ u = \cot\phi;$$
$$\xi = \int_{\phi}^{\frac{1}{2}\pi} \tau \cot\phi \csc\phi\, d\phi = \int_0^u u e^{u/45} du/(1+u^2)^{\frac{1}{2}};$$
$$\eta = \int_{\phi}^{\frac{1}{2}\pi} \tau \csc\phi\, d\phi = \int_0^u e^{u/45} du/(1+u^2)^{\frac{1}{2}};\ \sigma = 45(\tau - 1).$$

The tables are for $\phi = 1°(0°.1)12°.9$, $167°.1(0°.1)179°$, τ mostly to 4S; ξ mostly to 3–4S; $\pm\eta$ mostly 4S; $\pm\sigma$, 3–4S, η and σ — for $90° < \phi < 180°$. Also for $\phi = 10°(1°)170°$, all the 4 functions to 4D, except some terminal values to 5D.

434[F].—D. H. LEHMER, "On the factors of $2^n \pm 1$," Amer. Math. So., *Bull.*, v. 53, Feb. 1947, p. 164–167, 15.1 × 24 cm.

Professor Lehmer gives factors of $2^n - 1$, in 32 cases, for values of n from 113 to 489, and of $2^n + 1$ in 44 cases, for values of n from 91 to 500. This list for $n \leqslant 500$ was intended to supplement the fundamental table of CUNNINGHAM & WOODALL[1] and the addenda to this list found by KRAÏTCHIK.[2] It is believed that all factors under 10^6 have now been found, and that any other factors of $2^n - 1$ for $n \leqslant 300$, or of $2^n + 1$ for $n \leqslant 150$, lie beyond 4538800.

Eight complete factorizations, n varying from 91 to 170, are given; the fifth of these for $2^{123} + 1$ has been already noted in MTE **107**. The first and eighth correct errors in Kraïtchik and in Cunningham & Woodall.

Eleven of the new factors given by Lehmer pertain to Mersenne numbers $2^p - 1$, p a prime not greater than 257. These factors are included in the range $p = 113$ (now completely factored) to $p = 233$. Of the 55 Mersenne numbers 12 are prime ($p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127$), 14 are composite and completely factored, for 9 two or more prime factors are known, for 8 only one prime factor is known, 11 are composite but no factor known, and in one case ($p = 193$) the character is unknown. As indicated above, any other factor now discovered for a Mersenne number must be greater than 4538800.

Professor H. S. UHLER completed the proof that M_{199} was composite on July 27, 1946 (Amer. Math. Soc. *Bull.*, v. 53, 1947, p. 163–164); and that M_{227} was composite on June 4, 1947; see also *MTAC*, v. 1, p. 333 (M_{157}), 404 (M_{167}), v. 2, p. 94 (M_{229}). In the article here reviewed D. H. L. checked the last two results at which Uhler had arrived, by showing that M_{167} had the factor 2349023 and M_{229} the factor 1504073.

R. C. A.

[1] A. J. C. CUNNINGHAM & H. J. WOODALL, *Factorisation of* ($y^n \pm 1$), London, 1925.
[2] M. KRAÏTCHIK, (a) *Recherches sur la Théorie des Nombres*, v. 2, Paris, 1929; (b) "Factorisation de $2^n \pm 1$," *Sphinx*, v. 8, 1938, p. 148–150.

435[F].—Nils Pipping, "Goldbachsche Spaltungen der geraden Zahlen x für x = 60000–99998," Åbo, Finland, Akademi, *Acta, Math. et Phys.*, v. 12, no. 11, 1940, 18 p. 16 × 23.7 cm.

This table is an extension, for the range indicated in the title, of a previous table[1] for the range $x < 60000$. Its purpose is to verify the unproved Goldbach conjecture that every even number x greater than 4 is the sum of two odd primes. For the present range the conjecture is true with plenty to spare. Of the 20000 even numbers x in this range the author finds that 15315 of them are representable in such a way that the largest possible prime is involved: that is, one of the primes q in $x = p + q$ can be taken as the greatest prime not exceeding $x - 3$. The remaining 4685 numbers x are listed in the table together with the least prime m_x for which $x - m_x$ is also a prime. The largest m_x occurs at $x = 63274$ where m_x has the value 293.

D. H. L.

[1] N. Pipping, "Die Goldbachsche Vermutung und der Goldbach-Vinogradowsche Satz," Åbo, Finland, Akademi, *Acta, Math. et Phys.*, v. 11, no. 4, 1938, 25 p.

436[F].—Ernst S. Selmer & Gunnar Nesheim, "Tafel der Zwillingsprimzahlen bis 200.000," K. Norske Videnskabers Selskab, Trondhjem, *Forhandlinger*, v. 15, 1942, p. 95–98.

This is a short table of those values of n for which $6n + 1$ and $6n - 1$ are both primes less than 200,000. Since all prime pairs except (3, 5) are of this form we have in effect a table of prime pairs under 200,000. The number of these less than 100,000 was found to be 1224 which is in agreement with the count by Glaisher.[1] The number of prime pairs in the second 100,000 was found to be 936, which differs from the count 935 made by Sutton,[2] and Hardy & Littlewood.[3] The present list is based on the list of primes by Lehmer.[4]

D. H. L.

[1] J. W. L. Glaisher "On certain enumerations of primes," BAAS, *Report, 1878*, p. 470–471.
[2] C. S. Sutton, "An investigation of the average distribution of twin prime numbers," *Jn. Math. and Phys.*, v. 16, 1937, p. 41–42. See RMT **345**.
[3] G. H. Hardy & J. E. Littlewood, "Partitio numerorum III: On the expression of a number as a sum of primes," *Acta Math.*, v. 44, 1923, p. 44.
[4] D. N. Lehmer, *List of Primes from 1 to 10 006 721*. Washington, 1914.

437[F].—Ernst S. Selmer & Gunnar Nesheim, "Die Goldbachschen Zwillingsdarstellungen der durch 6 teilbaren Zahlen 196.302–196.596," K. Norske Videnskabers Selskab, Trondhjem, *Forhandlinger*, v. 15, 1942, p. 107–110.

This note contains a table (p. 108) giving the number of representations of $6n$ as a sum of two primes in which each prime is one of a pair of twin primes, for each integer of the form $6n$ between 196301 and 196597. Besides the actual count of such representations, approximate values are given as computed from the following formula of Stäckel

$$4.1532\cdot[\pi(6n)]^4(6n)^{-3}\prod_p \frac{p-2}{p-4}\prod_q \frac{q-3}{q-4}$$

where $\pi(x)$ is the number of all primes $\leqslant x$ and, in the two products, p ranges over the prime factors > 3 of n, while q ranges over the odd prime factors of $3n \pm 1$. The ratio of this approximation to the exact count is also tabulated. This ratio ranges from .83 to 1.25 but has an average of 1.001. The approximate and exact values are compared graphically. The exact count was based on the authors' previous table (RMT **436**) of prime pairs.

D. H. L.

438[F].—Dov YARDEN,[1] "Luach Mispare Fibonatsi" [Table of Fibonacci numbers], *Riveon Lematematika Lelimud Vlemechkar* [Quarterly Journal of Mathematics for Study and Research], Jerusalem, Palestine, v. 1, no. 2, Sept. 1946, p. 35–37. 21.6 × 33.6 cm. The text of this periodical, edited by Dov Yarden for graduate students of the University of Jerusalem, is entirely in Hebrew, and mimeographed on only one side of each sheet. The cover title-page is printed.

This table gives the two Fibonacci sequences U_n, V_n for $n = 0(1)128$. These famous numbers are defined by the recurrence formulae

$$U_n = U_{n-1} + U_{n-2}, \quad U_0 = 0, \quad U_1 = 1$$
$$V_n = V_{n-1} + V_{n-2} \quad V_0 = 2, \quad V_1 = 1$$

and have an extensive literature. Besides the mere values of these numbers the table gives their factorization into primes. These are complete as far as U_{66} and V_{61}. Beyond these points many entries have large factors enclosed in parentheses, indicating numbers of unknown composition, while many others are completely factored. The author was unaware of the work of POULET and KRAÏTCHIK summarized in Kraïtchik's table.[2] However certain small factors appear in Yarden's table which are not given in that of Kraïtchik as follows:

n	Factor of V_n	n	Factor of V_n
94	563	114	227
101	809	119	239
103	619	124	743
106	1483	127	509
112	223·449		

Addenda to the present table are promised for a future issue.

D. H. L.

EDITORIAL NOTES: Primitive factors in U_n, V_n, that is, the product of those factors which have not occurred previously, are underlined. The statement concerning $n = 106$ was in a communication from the author to the reviewer. There are the following errors in the table: U_{87}, the factor 514229 should not be underlined, since it occurs previously in U_{29}; U_{122} first factor, for first figure 3, *read* 2; V_{57}, *for* 9343, *read* 9349; V_{117}, *for* 67861, *read* 79·859; V_{120}, last factor, *for* first figure 3, *read* 2.

[1] Library of Congress transliteration is here, and later, employed. In *Scripta Mathematica*, v. 11, 1945, the name occurs as Dov Juzuk.

[2] M. KRAÏTCHIK, *Recherches sur la Théorie des Nombres*. v. 1, Paris, 1924, p. 77–81.

439[F].—D. YARDEN, "Luach Tsiyune-hehofa'a Besidrath Fibonatsi" [Table of the ranks of apparition in Fibonacci's sequence], *Riveon Lematematika*, v. 1, no. 3, Dec. 1946, p. 54. 21.6 × 33.6 cm. Compare RMT **438**.

The notion of the rank of apparition of a prime in Fibonacci's sequence

$$U_0 = 0, \quad U_1 = 1, \quad U_2 = 1, \quad U_3 = 2, \quad \cdots, \quad U_{n+1} = U_n + U_{n-1}$$

is the counterpart of the exponent of a number with respect to a prime modulus; it is simply the least positive subscript n for which U_n is divisible by the given prime p. The author denotes this function of p by $a(p)$. Thus $a(2) = 3$. By a theorem of Lucas, $a(p)$, for p an odd prime, is some (unpredictable) divisor of $p - \epsilon$, where $\epsilon = (5/p)$ is Legendre's symbol and has the value

$$\epsilon = \begin{cases} +1 & \text{if} \quad p = 10k \pm 1 \\ 0 & \text{if} \quad p = 5 \\ -1 & \text{if} \quad p = 10k \pm 3 \end{cases}$$

The present note gives a table of $a(p)$ for all primes $p \leq 1511$. Besides $a(p)$ the number $p - \epsilon$ is given in factored form. In those cases where $a(p) < p - \epsilon$, the factors of the "residue index" $(p - \epsilon)/a(p)$ are underlined. For every prime p the Fibonacci numbers are periodic, modulo p. The proper period is $4a(p)$ if $a(p)$ is odd, $2a(p)$ if $a(p)$ is divisible by 4, and $a(p)$

otherwise. This appears to be the first table of its kind. It gives indirectly all small prime factors of all Fibonacci numbers, where by "small" we mean $\leqslant$ 1511.

D. H. L.

EDITORIAL NOTE: Comparing this table with a short one for $p < 1000$ in M. KRAÏTCHIK, *Recherches sur la Théorie des Nombres*, v. 1, Paris, 1924, p. 55, we note three errors in Kraïtchik: p. 269, *for* $\gamma = 1$, *read* $\gamma = 4$; p. 499, *for* $\gamma = 2$, *read* $\gamma = 1$; p. 743, *for* $\gamma = 1$, *read* $\gamma = 3$. In Yarden's table, column of p's, there is a misprint of 367 for 467.

440[L].—HARVARD UNIVERSITY, Computation Laboratory, *Annals*, v. 5: *Tables of Bessel Functions of the First Kind of Orders Four, Five, and Six*; v. 6: *Tables of Bessel Functions of the First Kind of Orders Seven, Eight, and Nine*. By the staff of the Laboratory, Professor H. H. AIKEN, Technical Director, Cambridge, Mass., Harvard Univ. Press, 1947, xii, 650 p. and viii, 646 p. 19.5 × 26.7 cm. \$10.00 + \$10.00. Compare *MTAC*, v. 2, 176f, 185f, 261f. The offset printing continues to be of outstanding excellence.

The tabulation of these functions was undertaken at the request of the Bureau of Ships, in behalf of the Naval Research Laboratory, and continued under the cognizance of the Bureau of Ordnance. The tables were computed by recurrence from the values of $J_2(x)$ and $J_3(x)$ previously published in v. 4 of the *Annals*. In v. 5, the preface is by H. H. AIKEN and the pages ix–xi, on "Interpolation in the tables," with two numerical examples, are by R. M. BLOCH. The range of the parameter is x = [.012(.001)25(.01)99.99; 10D]. Since to 10D the values of $J_7(x)$ are zero for $x < .229$, it is with this value that the tabulation of v. 6 commences. Up to $J_7(.267)$ the values are all the same, .00000 00001.

The results in these volumes are almost entirely new. Among published tables, to at least 10D and for $n = 4(1)9$, are those by MEISSEL, for x = [0(1)24; 18D]; by AIREY, for x = [6.5(.5)16; 10D]; and by HAYASHI, for 15 values of x to at least 21D.

R. C. A.

441[L].—H. G. HAY, assisted by Miss N. GAMBLE and approved by G. G. MACFARLANE, *Five-Figure Table of the Function* $\int_0^\infty e^{-zy}\mathrm{Ai}^2(y - j_1)dy$ *in the Complex Plane*. (Great Britain, The Mathematical Group, Telecommunications Research Establishment (TRE), Malvern, Worcestershire), no. T 2047, November 1946, 22 p. 20.2 × 33 cm. Mimeographed. Compare *MTAC*, v. 2, p. 41, and RMT **444**. This publication is not generally available.

The need for these tables arose in connection with the determination of the eigenvalues of the wave equation for centimetric wave propagation in the atmosphere. The wave equation is

$$\frac{d^2U}{ds^2} + [s + \sum_{n=1}^{r} A_n e^{-\alpha_n s} + D_1]U = 0$$

in which the constants A_n and α_n are known. The equations from which the eigenvalue D_1 is found involve the function $F(z) = \int_0^\infty e^{-zy}Ai^2(y - j_1)dy$, and its first derivative, where $-j_1(= -2.3381)$ is the first zero of the Airy function Ai(y), as defined by MILLER[1] and tabulated in the complex plane by P. M. Woodward & Mrs. A. M. Woodward.[2] The function has also been discussed in the same connection by C. L. PEKERIS.[3] It satisfies the differential equation

$$dF/dz = (2z)^{-1}[\mathrm{Ai}'(-j_1)]^2 - [(2z)^{-1} + j_1 + \tfrac{1}{4}z^2]F,$$

from which the derivative can be calculated when $F(z)$ is known. Asymptotic formulae for F and F' can be used to extend the table to $|z| > 4$, by the methods given by PEKERIS.[3]

The tables giving values of the real and imaginary parts of $F(z)$ and $F'(z)$, to 5D, 16 p.,

cover the region $x = 0(.2)4$, $y = 0(.2)3.2$, in the upper half of the complex plane. The values of $\frac{1}{2}(\Delta_x^2 - \Delta_y^2)$ and $\Delta_x^2\Delta_y^2$ are placed beneath each function value, as in the Woodward tables, for applying their method of bivariate interpolation for a function of a complex variable.

In the computation of the tables each function value was calculated to six figures and the power series expansions around two enclosing origins were used as a first check. At this stage the error did not exceed five units in the last decimal place. The values were then rounded off by the Woodward method, which probably reduces error to less than a unit in the sixth decimal place. The tables presented in five-figure form are hoped to be correct to the number of figures shown. An error of one unit in the fifth figure may occur where the sixth figure is in the region of five.

Extracts from the introductory text

[1] J. C. P. MILLER, *The Airy Integral*, Cambridge, 1946 (see *MTAC*, RMT, **413**), and *MTAC*, v. 1, no. 7, p. 236.
[2] P. M. WOODWARD & Mrs. A. M. WOODWARD, "Four-figure tables of the Airy function in the complex plane," *Phil. Mag.*, s.7, v. 37, p. 236–261 (see *MTAC*, RMT **420**).
[3] C. L. PEKERIS, "Perturbation theory of the normal modes for an exponential *M*-curve in non-standard propagation of microwaves," *Jn. Applied Physics*, v. 17, 1946, p. 678.

442[L].—RAY S. HOYT, "Probability functions for the modulus and angle of the normal complex variate," *Bell System Technical Jn.*, v. 26, April 1947, p. 318–359. 15 × 22.8 cm.

There is a table on p. 346, of $Q(R) = 2\int_0^R t^{\frac{1}{2}}e^{-t}I_0(bt)d\lambda$, $t = \lambda^2/(1 - b^2)$, for $R = .2(.2).8$, and of $Q^*(R) = 2u^{\frac{1}{2}}\lambda\int_0^r \lambda^{-3}e^{-u}I_0(bu)d\lambda$, $u = 1/[\lambda^2(1 - b^2)]$, for $R = r^{-1} = 1.6, 2$. For each of these integrals $b = 0, .3(.1)1, .95$. Corresponding to each of the values of R there are in the table four rows under the various values of b. In the first row of any set of four rows are the values of $Q(R)$ or $Q^*(R) = e^{-R^2} +$ values of $P_{b,0}$, given in a table on p. 53 of a paper by Hoyt in *Bell System Technical Journal*, v. 12, 1933. In the second row are the values computed from formulae indicated above, to 4 or 5D. The third row of each set of four rows gives the deviations of the second row from the first row; and the fourth row expresses these deviations as percentages of the values in the first row.

Extracts from the text

443[L].—L. INFELD, V. G. SMITH & W. Z. CHIEN, "On some series of Bessel functions," *Jn. Math. Phys.*, v. 26, April 1947, p. 22–28.

"In a study of radiation from a cylindrical antenna in a rectangular wave guide we were confronted with the series $\sum_{m=0}^{\infty}(-1)^m Y_0(mx)$." Let $S = \sum_{m=1}^{\infty}(-1)^m Y_0(m\pi x)$. On p. 27 is a table of S, $x = [0(.2)3; 10D]$ rounded off from 13D calculations. There are also three 10D tables of six entries each to make possible interpolation near a discontinuity; these tables are of the functions (i) $S + \pi^{-1}\ln x$, (ii) $S + 2\pi^{-1}(1 - x^2)^{-\frac{1}{2}}$, (iii) $S + 2\pi^{-1}(9 - x^2)^{-\frac{1}{2}}$. On p. 26 are graphs of these four functions.

444[L].—G. G. MACFARLANE, *The Application of a Variational Method to the Calculation of Radio Wave Propagation Curves for an arbitrary Refractive Index Profile in the Atmosphere.* (Great Britain, The Mathematical Group, TRE, Malvern, Worces., no. T. 2048.) December, 1946. 15 p. + 5 plates of figures. 20.2 × 33 cm. This publication is not generally available.

The basis of discussion here is the differential equation already noted in RMT **441**. On p. 14 are two small tables. Denoting the zeros of the Airy function by j_r, then for $r = 1(1)10$ are given the values of j_r to 5D and of $\mathrm{Ai}'(-j_r)$ and $[\mathrm{Ai}'(-j_r)]^2$, each to 6D. The other table, to 6D, is of $P_{rs} = \mathrm{Ai}'(-j_r)\mathrm{Ai}'(-j_s)/(j_r - j_s)^2$, $r > s$, $P_{rr} = (j_r/3)[\mathrm{Ai}'(-j_r)]^2$, $r = 1(1)5$, $s = 1(1)5$.

445[L].—F. W. J. Olver, "Note on a paper of H. Bateman," *Jn. Appl. Phys.*, v. 17, Dec. 1946, p. 1127. 19 × 26.1 cm. See RMT **308**, v. 2, p. 126.

The text of the Note: "In a paper [*Jn. Appl. Physics*, v. 17, 1946, p. 91–102] entitled 'Some Integral Equations of Potential Theory' the late Professor Bateman includes a table of $P_n(1 - 2e^{-t})$, to 15D, for $n = 1(1)10$, $t = 1(1)20$. A number of errors were noticed in this table and subsequent investigation showed the need for complete recomputation. This has been done and the new table, in which the values given are correct to within 0.52 units of the fifteenth decimal, is appended."

"Attention is also drawn to the erroneous value given [Reference 1, p. 99] for $P_4(u)$, where $u = 1 - 2e^{-\pi}$; the last four decimals given should read . . . 7662 instead of . . . 5162."

446[L].—Max Eric Reissner, "Stresses and small displacements of shallow spherical shells. II," *Jn. Math. Physics*, v. 25, Jan. 1947, p. 279–300. 17.4 × 25.1 cm. Table on p. 298.

For $\lambda = .1, .2, .5, 1(1)10$, are given 4S values for $Wr(\lambda)$, $Wu(\lambda)$, C_1 and C_2($k = 0$, $k = \infty$ and $\nu = \frac{1}{4}$), and 3D values for $f_1(\lambda)$ and $f_2(\lambda)$, ($k = 0$, $k = \infty$ and $\nu = \frac{1}{4}$), also with values for $\lambda = 0$.

$Wr(\lambda) = \mathrm{ber}\,\lambda\,\mathrm{kei}'\,\lambda - \mathrm{ber}'\,\lambda\,\mathrm{kei}\,\lambda$, $Wu(\lambda) = \mathrm{bei}\,\lambda\,\mathrm{kei}'\,\lambda - \mathrm{bei}'\,\lambda\,\mathrm{kei}\,\lambda - \frac{1}{2}\lambda$,
$f_1(\lambda) = (8/\lambda^2)[\frac{1}{4}\pi + \mathrm{kei}\,\lambda + C_1(\mathrm{ber}\,\lambda - 1) + C_2\,\mathrm{bei}\,\lambda]$,
$f_2(\lambda) = 2[-\,\mathrm{ker}\,\lambda + C_1\,\mathrm{bei}\,\lambda - C_2\,\mathrm{ber}\,\lambda]$.
For $k = 0$, $C_1 = [Vu(\lambda) + \mathrm{bei}'\,\lambda/\lambda]/Vb(\lambda)$, $-\,C_2 = [Vr(\lambda) + \mathrm{ber}'\,\lambda/\lambda]/Vb(\lambda)$;
For $k = \infty, \nu = \frac{1}{4}$, $C_1 = \{\frac{3}{4}[Vu(\lambda) + \mathrm{bei}'\,\lambda/\lambda] - \lambda Wu(\lambda) - \frac{1}{2}\}/[\frac{3}{4}Vb(\lambda) - \lambda Wb(\lambda)]$,
$-\,C_2 = \{\frac{3}{4}[Vr(\lambda) + \mathrm{ber}'\,\lambda/\lambda] - \lambda Wr(\lambda)\}/[\frac{3}{4}Vb(\lambda) - \lambda Wb(\lambda)]$;
$Vb(\lambda) = (\mathrm{ber}'\,\lambda)^2 + (\mathrm{bei}'\,\lambda)^2$, $Vr(\lambda) = \mathrm{ber}'\,\lambda\,\mathrm{ker}'\,\lambda + \mathrm{bei}'\,\lambda\,\mathrm{kei}'\,\lambda$,
$Vu(\lambda) = \mathrm{bei}'\,\lambda\,\mathrm{ker}'\,\lambda - \mathrm{ber}'\,\lambda\,\mathrm{kei}'\,\lambda$, $Wb(\lambda) = \mathrm{ber}\,\lambda\,\mathrm{bei}'\,\lambda - \mathrm{bei}\,\lambda\,\mathrm{ber}'\,\lambda$.

The calculations were "carried out with two more decimals than are listed in the table." The author informed us that c_1, c_2 in the table should be C_1, C_2.

447[L, M].—H. B. Dwight, *Tables of Integrals and Other Mathematical Data.* Revised ed., New York, Macmillan, 1947. x, 250 p. 13.4 × 20.3 cm. $2.50.

The first edition of these tables has been already reviewed in *MTAC*, v. 1, p. 190–191, and errata therein listed, p. 195–196. Although 28 pages have been here added, the main numbering of different items remains unchanged except that a probability integral table (no. 1045) has been introduced. Otherwise the numerical tables in the Appendix are unchanged, except for corrections. All the errors which we previously noted have been expunged. The considerable number of items added in other parts of the volume includes groups of integrals involving $(a^2 + bx + c)^{\frac{1}{2}}$, $(a + b\sin x)^{-1}$, and $(a + b\cos x)^{-1}$, and also material on inverse functions of complex quantities and on Bessel functions. To the 37 volumes listed under "References" in the first edition, 30 new works have been added, but none dated later than 1945. Scattered throughout the book are constant directions for consultation of this list. For example, on p. 177: "For tables of $K_0(x)$ and $K_1(x)$, see Ref. 50, p. 266, and Ref. 12, p. 313. Tables of $e^xK_0(x)$ and $e^xK_1(x)$, Ref. 13. Tables of $(2/\pi)K_0(x)$ and $(2/\pi)K_1(x)$, Ref. 17." There are some other literature lists in footnotes; the old one mentioning *Reports* of B.A.A.S. still has (p. 241) 1916, "p. 109," when I fancy that p. 122 is intended. On p. 129, T. 1045 is referred to as T. 1035. In 808.4 *read* $-J_n'(x)$; in 812.4, $-N_n'(x)$. The revised edition of this excellent work may be heartily recommended.

R. C. A.

448[U].—Great Britain, Nautical Almanac Office, *Astronomical Navigation Tables*, v. *Q*, *Latitudes N70°–N79°*. (*British Air Publication* no. 1618.) London, His Majesty's Stationery Office, [1944], iv, 341 p. 16.3 × 24.6 cm.

This volume is the fifteenth of a series, the first fourteen of which were reproduced by photolithography in the United States as *H. O. 218*. It is understood that this volume is not to be reproduced as a volume of *H. O. 218*.

The latter have been reviewed earlier (*MTAC*, v. 1, p. 82f.) and the reader is referred there for the general description of the tables, methods of use, etc. This review will concentrate on the points at which this volume differs from the earlier ones.

Each of the first fourteen volumes covered only five degrees of latitude, but could be used in the appropriate belt either north or south of the equator. Only twelve bright stars are offered in this volume instead of twenty-two, namely Aldebaran, Alpheratz, Altair, Arcturus, Betelgeuse, Capella, Deneb, Dubhe, Pollux, Procyon, Regulus, and Vega. The tabular material for these stars covers only the northern latitude 69°30′ to 79°30′, but the general tables for sun, moon and planets, declinations 0(1°)23°, can be used in either hemisphere.

The lower limit of altitude tabulated is 1° for declinations up to 16°, 10° for other declinations. This may be compared to the uniform lower limit of 10° in the earlier volumes. When one recalls that, for weeks at a time in the polar regions, the sun may not reach an altitude as great as 10°, one senses the importance of this change. It is to be hoped that the refraction corrections were also adjusted for the polar regions, since at low altitudes, the difference in refraction corrections between temperate and polar temperatures may amount to more than 2′.

The star tables shortly to appear as Hydrographic Office *Publication No. 249* will serve the same purpose as the star tables in this volume and will have the added advantage that they will be useful over a wider range of latitudes. There will still remain a need for a Hydrographic Office publication to cover the sun, moon and planets for aerial navigation in polar latitudes. The volumes of *H. O. 214* (*MTAC*, v. 1, p. 81f) cover the latitudes and the declinations but were designed primarily for surface navigation. Aerial navigation needs a single book of smaller bulk and weight and faster to use than the volumes of *H. O. 214* required for polar travel.

CHARLES H. SMILEY

Brown University

449[V].—A. HORACE WILLIAMS KING, *Manning Formula Tables for Solving Hydraulic Problems*. v. 1, *Flow in Pipes* . . .; v. 2, *Flow in Open Channels* . . ., New York and London, McGraw-Hill, 1937–1939. xi, 351 p.; xiv, 379 p. 15 × 22.5 cm.

B. SHERMAN M. WOODWARD & CHESLEY J. POSEY, *Hydraulics of Steady Flow in Open Channels*, New York, Wiley, 1941; Tables, p. 8–14. 14.7 × 23 cm.

C. JOSE S. GANDOLFO, "Calculo de canales en movimiento uniforme. Conversión de las tablas de Woodward y Posey en adimensionales," *Revista de la Administración Nacional del Agua*, Buenos Aires, Argentina, v. 9, Dec. 1945, p. 449–458. 19.5 × 28.3 cm.

The first algebraic expression concerning the flow of water in pipes or open channels was published in 1757 by ANTOINE DE CHEZY (1718–1798). It is the well-known Chezy formula, $V = C\sqrt{Rs}$, where V is the mean velocity of flow, R is the hydraulic radius, or ratio of the area of the cross-section to the length of the wetted perimeter of the cross-section, s is the slope of the hydraulic gradient and C is a numerical coefficient which varies with the roughness of the channel lining and with the hydraulic radius. Many experimenters have proposed formulae for the evaluation of the coefficient C.

In 1869, GANGUILLET & KUTTER, Swiss engineers, after a great deal of experimentation and measurement on both natural and artificial channels, suggested a rather complicated formula in which not only n, a coefficient of roughness, and R, but also s appeared. They also gave a list of various types of channel linings, with the appropriate value of n in each case. Their formula for the determination of C has had and is still having considerable use, both in the United States and abroad, although subsequent investigators do not agree that the value of the slope has any bearing. In 1890, ROBERT MANNING, an Irish engineer, one time

president of the Institution of Civil Engineers of Ireland, presented a formula[1] for the determination of C in terms of R and n, with values of n as proposed by Ganguillet and Kutter. His formula is

$$C = R^{1/6}/n$$

and this value inserted in the Chezy formula gives:

$$V = R^{\frac{2}{3}}s^{\frac{1}{2}}/n$$

in metric units, or

$$v = 1.486R^{\frac{2}{3}}s^{\frac{1}{2}}/n$$

for use with units of feet and seconds.

The commonly used form of the Manning formula is:

$$Q = 1.486aR^{\frac{2}{3}}s^{\frac{1}{2}}/n,$$

where Q is the discharge in cubic feet per second and a is the cross-sectional area of the channel in square feet. This formula, as applied to both pipes and open channels, enjoys wide-spread use. Various tables have been prepared to facilitate its use.

A. For a circular pipe, flowing full, the hydraulic radius, or R, is equal to the area of the circular cross-section divided by its circumference, or $d/4$, where d is the internal diameter of the pipe. Replacing R in the Manning formula by $d/4$ and solving the equation for d, there results:

$$d_i = \left(\frac{1630\,Qn}{s^{\frac{1}{2}}}\right)^{3/8},$$

where d_i is the internal diameter of the pipe in inches. The tables in v. 1, p. 1–351, give values of d_i for various combinations of Q, n and s. Q is given in cubic feet per second and also in equivalent gallons per minute or million gallons per day. With any three of the quantities in the formula known, the fourth one can be found from the tables.

Q = .001(.0005).005(.001).02(.005).05(.01).2(.5)1(.1)5(.5)20(1)100(5)200(10)500(25)1000 (50)2000(100)5000(250)10000,
s = [.00002(.00002).0002(.00005).0005(.0001).001(.0002).003(.0005).005(.001).01(.002) .02(.005).05; 2–4S],
n = .01(.001).02.

For trapezoidal channels, including those with rectangular and triangular sections, having a depth of flow D, a bottom width b and side slopes of z horizontal to 1 vertical, for circular channels having a depth of flow D and a diameter d, and for parabolic channels having a depth of flow D and a top width T, the Manning formula may be put in the form

$$Q = KD^{8/3}s^{\frac{1}{2}}/n,$$

where K is a function of z and the ratio of D to b for trapezoidal sections, a function of the ratio of D to d for circular sections and a function of the ratio D to T for parabolic sections. These respective values of K are given in Tables A, C and E of v. 2.

Table A, p. 352–365, D/b = [.001(.001).5(.01)2, ∞; 3S], $z = 0(\frac{1}{4})1(\frac{1}{2})3, 4$.
Table C, p. 378, D/d = .01(.01)1.
Table E, p. 379, D/T = .01(.01)1.

The Manning formula may also be written:

$$Q = K'(b,\ d \text{ or } T)^{8/3}s^{\frac{1}{2}}/n$$

for trapezoidal, circular or parabolic channels, respectively, where K' is a different function of the same variables which combine to form K. These respective values of K' are given in Tables B, D and F of v. 2. Table B does not, however, contain values of K' for triangular channels.

Table B, p. 365–377, D/b = .001(.001).5(.01)2, $z = 0(\frac{1}{4})1(\frac{1}{2})3$, 4
Table D, p. 378, D/d = .01(.01)1
Table F, p. 379, D/T = .01(.01)1.

Tables B, D and F are really superfluous as they simply provide alternate methods for securing the same results that tables A, C and E give.

The Manning formula may be put in the general form

$$Q = FL^{8/3}s^{1/2}/n,$$

where F is either K or K' and L is a linear dimension, either D, b, d or T. The main table in v. 2, p. 1–351, gives a solution of this equation for L, transformed to the form

$$L = \left(\frac{Qn}{F}\right)^{3/8}\left(\frac{1}{s}\right)^{3/16};$$

Qn/F = .00005(.00005).0002(.0001).0004(.0002).001(.0005).005(.001).01(.002).02(.005) .08(.01).3(.02).7(.05)2(.1)5(.2)10(.5)20(1)40(2)80(5)150(10)300(20)600(50)1500(100) 3000(200)5000(500)10000(1000)20000(2000)40000(5000)100000,
s = [0(.000001).0001(.00001) 001(.0001).01(.001).1(.01)1; 3–4S].

In H. W. KING, *Handbook of Hydraulics*, third ed., second impression, 1939, p. 331–358, (10 × 16.7 cm.) there are tables of (a) K and K' for trapezoidal channels; (b) values of $1/K'^2$ in the formula $s = \left(\frac{Qn}{K'b^{8/3}}\right)^2$, for trapezoidal channels; (c) values of K and K' for circular and parabolic channels.

B. The tables here are less detailed and of somewhat different range.

Table 102A, K' for trapezoidal sections, p. 8–9,
D/b = .02(.01).5(.02)1(.05)2(.1)3(.2)4(.5)5; z = [0(¼)1½(½)3, 4; 3S].
Similar to Table B in **A.**

Table 102B, K for trapezoidal sections, p. 10–11,
D/b = .01(.01).5(.02)1(.05)2(.1)3(.2)4(.5)5, z = 0(¼)1½(½)3, 4.
Similar to Table A in **A.**

Table 103, for circular sections, p. 12, d/D = .01(.01)1.

In the fourth and fifth columns of this table are the values of K and K' which are given, respectively, in Tables C and D of **A.** In the second and third columns are shown, respectively, values of the ratio of the area of flow to the square of the diameter of the section and of the ratio of the hydraulic radius to the diameter of the section.

Tables 104A and 104B for special sections, p. 13–14,
D/r = .02(.02)1(.05)2(.1)3, z = ½(¼)1(½)2.

For special round-bottomed channels, of radius r, with sides tangent to the bottoms and having slopes of z horizontal to 1 vertical, and with depth of flow D, the Manning formula may be put in the forms:

$$Q = Kr^{8/3}s^{1/2}/n, \qquad \text{and} \qquad Q = K'D^{8/3}s^{1/2}/n,$$

where K and K' are functions of z and of the ratio of D to r. Values of K and K' are given in Tables 104A and 104B, respectively.

C. These tables are rather trivial, being tables of **B**, with tabular values of K and K' divided by 1.486, for use with metric units.

Table No. 1 for trapezoidal section,
D/b = .01(.01).5(.02)1(.05)2(.1)3(.2)4(.5)5, z = 0(¼)1½(½)3, 4.

Table No. 2 for trapezoidal sections,
D/b = .02(.01).5(.02)1(.05)2(.1)3(.2)4(.5)5, z = 0(¼)1½(½)3, 4.

Table No. 3 for circular sections,
D/d = .01(.01)1.

Table No. 4 for special sections,
D/r = .02(.02)1(.05)2(.1)3, z = ½(¼)1(½)2.

FRANKLIN O. ROSE

Brown University

[1] R. MANNING, "Flow of water in open channels and pipes," Institute of Civil Engineers of Ireland, *Trans.*, v. 20, 1890.

EDITORIAL NOTE: There is a "Nomograph for solving Manning's formula," by PAUL McH. ALBERT, in *Water Works & Sewerage*, v. 92, 1945, p. R278–279.

MATHEMATICAL TABLES—ERRATA

References have been made to Errata in the article "On a Scarce Factor Table" (Beeger, Chernac, Legendre); and in RMT **428** (Boll, De Morgan), **429** (Vuagnat), **432** (Lambert), **434** (Kraïtchik, Cunningham & Woodall), **438** (Yarden), **439** (Kraïtchik, Yarden), **445** (Bateman), **446** (Reissner), **447** (Dwight); UMT **60** (Thomson); N **75** (Pal).

110. A. N. Dinnik. *Tablifsi Besselevikh Funkfsiĭ drobovogo porîadku* [Tables of Bessel Functions of fractional order], Vseukrain'ska Akad. N., *Prirodnicho-tekhnichniĭ viddil*, Kiev. 1933. Compare *MTAC*, v. 1, p. 287.

On p. 23 Dinnik gives the first 5 zeros of $J_{\pm 5/6}(x)$, $J_{\pm 1/6}(x)$ to 3D. The following errors of more than a unit in the last place were found by recomputation, either by interpolation or from the series for $J_\nu(x)$: first zero, $J_{-5/6}(x)$, *for* 0.844, *read* 0.849; second zero, $J_{-5/6}(x)$, *for* 4.736, *read* 4.136; second zero, $J_{-1/6}(x)$, *for* 5.344, *read* 5.257; fifth zero $J_{-1/6}(x)$ *for* 14.618, *read* 14.668; fifth zero, $J_{1/6}(x)$, *for* 15.184, *read* 15.192; fifth zero, $J_{5/6}(x)$, *for* 16.202, *read* 16.218.

NBSMTP

Editorial Note: In Novocherkask, Donskoĭ Politekh. Institut, *Izvestiîa*, v. 2, 1913, p. 356, Dinnik gives the correct value, 5.26, for the second zero of $J_{-1/6}(x)$.

111. FMR, *An Index of Mathematical Tables*. 1946. See *MTAC*, v. 2, p. 13–18, 136, 178–181, 219–220, 277–278.

On p. 235, section 16.11 $P_n(x)$, line 3, "$n = 20$" should replace "$n = 18$" in the sentence "The first forms are given to $n = 18$ in Prévost 1933." In fact, Prévost gives the exact algebraic expressions for $P_0(u)$, $\cdots$, $P_{20}(u)$ on p. 156 and also the Fourier series for $P_0(\cos\theta)$, $\cdots$, $P_{18}(\cos\theta)$, on p. 157. This latter fact should have been noted on p. 236, section 16.21 $P_n(\cos\theta)$.

This information is given correctly in NBSMTP, *Tables of Associated Legendre Functions*, 1945, p. xli, item 28. I checked Egersdörfer's expressions for $P_n(u)$, p. xxxvii, item 10, by comparison with Prévost's up to $n = 20$ and found that they agree.

Herbert E. Salzer

NBSMTP

112. P. R. E. Jahnke & F. Emde, *Tables of Functions with Formulae and Curves*, 1933 (p. 319), all later editions p. 269. See *MTAC*, v. 1, p. 391f; v. 2, p. 47.

The error found in line − 5, on the page thus referred to, occurs in the numerical coefficient of the term containing $1/n^{z+6}$ in the expansion of $\zeta(z)$; *for* 1/3024, *read* 1/30240. This particular coefficient is equal to $B_3/6!$, the general value being $B_r/(2r)!$, where B_r is the rth Bernoulli number (see p. 322 or 272).

This error was noted by Dr. R. H. Healey and myself in *Amalgamated Wireless Australasia* (*A.W.A.*) *Technical Rev.*, v. 6, 1943, p. 134.

J. K. Mackenzie

National Standards Laboratory,
Chippendale, Sydney, N.S.W.,
Australia

113. Marchant Calculating Machine Co., *Cube Root Divisors*. Publication no. 68, 1944.

The review, *MTAC*, v. 1, p. 356, states that one application of the process will give the required root correct to at least 5S. This is a proper statement on the usual assumption that

the 5th-figure error will not exceed 1. However, this table, computed by the writer, states that the root differs from the true one by less than 5 in the sixth figure. Because 6-figure divisors are used which introduce a possible rounding error of 5 in 7th place, the resulting relative error at various points of the table may cause the calculated root to be in error by as much as 7 in 6th place if the divisors are taken from col. 3 for N of 192.5 or more. This is because the bound of error due to spacing of the N's in col. A is about 4 in 6th place for values close to the mid-points of col. A. A check of roots at such mid-points starting upwards from $N = 192.5$ shows that at $N = 242$ using as "nearest divisor" that for 244, the error of the root is 7 in 6th place; the divisor for 244 has a 7th-place error of 5. There may be other cases.

The table readily may be brought within the error-limit of 5 in 6th place by using 7-place divisors in col. 3 for arguments above 191. A computation of such 7-place divisors has been made by multiplying the constant $3 \times 100^{\frac{2}{3}}$ by the values of $x^{\frac{2}{3}}$ taken from NBSMTP, *Tables of Fractional Powers* (see *MTAC*, v. 2, p. 205), for x equal to the arguments of col. A $\times 10^{-2}$. This recalculation shows no errors of the original amounts. The desired 7th-figure terminals will be supplied upon application to the writer at 2254 Bancroft Way, Berkeley 4, Calif.

TRACY W. SIMPSON

2254 Bancroft Way,
Berkeley, California

114. J. de MENDIZÁBAL TAMBORREL, *Tables des Logarithmes à Huit Décimales des Nombres de 1 à 125000*···. Paris, 1891.

The 10,000 logarithms from 100,000 to 110,000 in the above tables were compared with the proof of a table of 8-figure logarithms appearing in a new table now in press for Messrs. Chambers. The only error found was the following:

log 101597 *for* 8083, *read* 8088.

L. J. C.

EDITORIAL NOTE: Since Mendizábal Tamborrel tells us that he copied his logarithms of the numbers 100000 to 108000 from the tables of H. L. F. Schrön, 1799–1875 (see *MTAC*, v. 1, p. 40) who gives the logarithm of this number correctly, the error is a case of miscopying. It may be well to put on record here references to errata in Schrön's tables (of which there were many editions after the first in 1860) as listed in *Archiv d. Math. Phys.* These are as follows: v. 34, 1860, p. 368; v. 35, 1860, p. 120; v. 36, 1861, p. 384; v. 41, 1864, p. 120, 240, 496; v. 43, 1865, p. 120, 244; v. 45, 1866, p. 236; v. 46, 1866, p. 360; v. 47, 1867, p. 120, 362; v. 51, 1870, p. 128.

115. L. M. MILNE-THOMSON & L. J. COMRIE, *Standard Four-Figure Mathematical Tables*. London, 1931. See *MTAC*, v. 1, p. 16, 84, 95, 192, 335, 432.

The following end-figure errors of exactly a unit each were found in the course of preparing a new 6-figure edition of Chambers' *Mathematical Tables*. On page 196:

	$\sec^{-1}x$		$\operatorname{cosec}^{-1}x$	
x	*For*	*Read*	*For*	*Read*
1.72	.9505.	.9504.	.6203·	.6204·
1.73	.9546.	.9545.	.6162·	.6163·
1.74	.9586·	.9585·	.6122.	.6123.

These errors occurred in the course of building up 7-figure values of these functions from their first differences; two compensating errors of 1000 in the seventh decimal (1 in the fourth decimal) were introduced. As $\operatorname{cosec}^{-1}x = \frac{1}{2}\pi - \sec^{-1}x$, both functions were affected. Such errors could not occur with the use (introduced since these tables were published) of the National machine for differencing and subtabulation. No other errors are known in these tables.

L. J. C.

116. NBSMTP, *Table of Natural Logarithms*, v. 4, 1941.

P. 462, argument 9.6061, *for* 2.26239 48763 487638, *read* 2.26239 83133 487638.

W. F. SHENTON

American University,
Washington, D. C.

EDITORIAL NOTES: Dr. Lowan reported that every surmise as to the cause of this serious error has had no real basis in the evidence at hand.

With regard to the error noted in MTE 109, p. 168, where 5.33452 58209 12879 must replace 5.33452 58202 12879, Dr. Lowan wrote as follows: "The curious thing about this error is that both the original manuscript page 168 and the corresponding negative are correct. Moreover in two of the fifteen copies of the table of exponential functions available at the NBSMTP the correct though faint digit 9 appears in the group 58209. It thus appears that an extremely careless worker tampered with the plate, converting the 9 to a 2. It is fortunate from our point of view that in making this change, the new figure 2 is conspicuously different from the typed 2; it is precisely this fact which made it possible for us to discover this error."

Concerning the error on p. 304, MTE 109, Dr. Lowan wrote: "The original manuscript is correct, although the fourth digit, 9, is rather faint. In the process of retouching the negative the faint digit was changed to an 8 by a careless worker who did not deem it necessary to consult the manuscript page!".

117. *A Webb & Airey—Adams—Bateman—Olsson Error.*

If α and γ are arbitrary constants, which may be complex, in the equation

$$xy'' + (\gamma - x)y' - \alpha y = 0 \tag{1}$$

the complete solution is

$$y = AM(\alpha, \gamma, x) + Bx^{1-\gamma}M(\alpha - \gamma + 1, 2 - \gamma. x)$$

where

$$M(\alpha, \gamma, x) = 1 + \frac{\alpha}{1(\gamma)}x + \frac{\alpha(\alpha+1)}{1(2)(\gamma)(\gamma+1)}x^2 + \cdots$$

except when γ is zero or a negative integer; this case can be excluded with no loss of generality. When γ is a positive integer the coefficient of B is either infinite or identical with the coefficient of A. In this case one of the solutions takes a form analogous to the second solution of Bessel's equation of integral order, and the complete solution of (1) is

$$\begin{aligned} y = {} & C\{\ln x + \psi(1-\alpha) - \psi(\gamma) - \psi(1)\}M(\alpha, \gamma, x) \\ & + C\sum_{n=0}^{\gamma-2}(-1)^{\gamma+n}\Gamma(\gamma)B(n+\alpha-\gamma+1, \gamma-n-1)x^{n+1-\gamma}/n! \\ & + C\left[\frac{\alpha x}{\gamma}\left(\frac{1}{\alpha} - \frac{1}{\gamma} - 1\right) + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{x^2}{2!}\left(\frac{1}{\alpha} + \frac{1}{\alpha+1} - \frac{1}{\gamma} - \frac{1}{\gamma+1} - 1 - \tfrac{1}{2}\right)\right. \\ & + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!}\left(\frac{1}{\alpha} + \frac{1}{\alpha+1} + \frac{1}{\alpha+2} - \frac{1}{\gamma} - \frac{1}{\gamma+1} - \frac{1}{\gamma+2}\right. \\ & \left.\left. - 1 - \tfrac{1}{2} - \tfrac{1}{3}\right) + \cdots \text{ to infinity}\right] \end{aligned} \tag{2}$$

where $\psi(\gamma) = \Gamma'(\gamma)/\Gamma(\gamma)$ is the psi function and $B(m, n)$ is the Beta function.

This result was obtained by W. J. ARCHIBALD, "The complete solution of the differential equation for the confluent hypergeometric function," *Phil. Mag.*, s. 7, v. 26, 1938, p. 415–419. An incorrect result was given by H. A. WEBB & JOHN R. AIREY, "The practical importance of the confluent hypergeometric function," *Phil. Mag.*, s. 6, v. 36, 1918, p. 132, since the terms in x^{-1}, x^{-2}, $\cdots x^{-\gamma+1}$ were omitted. This erroneous result was copied by

H. BATEMAN, *Partial Differential Equations of Mathematical Physics*. Cambridge University Press, 1932, p. 457, and by EDWIN P. ADAMS, *Smithsonian Mathematical Formulae and Tables of Elliptic Functions*. Washington, 1922 and 1939, p. 186. In the American edition of Bateman's work, New York, 1944, p. 457, he gave a corrected result, except that his arbitrary constant for the terminating series should not equal the constant for the logarithmic term.

R. GRAN OLSSON, "Tabellen der konfluenten hypergeometrischen Funktion erster und zweiter Art," *Ingenieur-Archiv*, vol. 8, 1937, p. 99–103, also p. 376–377, used the incorrect results of WEBB & AIREY to compute tables of the second solution for $\gamma = 2, 3$.

The corrected series for the Whittaker function was first given by R. STONELEY, "The transmission of Rayleigh waves in a heterogeneous medium," R.A.S., *Mo. Not., Geophys. Suppl.*, v. 3, 1934, p. 227, which he obtained by comparing the terms of the series for $W_{k,m}(z)$, when $2m$ is an integer, which was given by S. GOLDSTEIN, "Operational representation of Whittaker's confluent hypergeometric function and Weber's parabolic cylinder function," London Math. Soc., *Proc.*, s. 2, v. 34, 1932, p. 103–125, with the corrected Webb & Airey formula.

MURLAN S. CORRINGTON

Radio Corporation of America,
Camden, N. J.

UNPUBLISHED MATHEMATICAL TABLES

60[F].—JOHN THOMSON, *Table of Twelve-Figure Logarithms*, mss. in the possession of the Royal Astronomical Society. Compare UMT **30**, v. 1, p. 368.

This MS table is described at length by J. W. L. GLAISHER in R.A.S., *Mo. Not.*, v. 34, 1874, p. 447–475. Glaisher lists many errors found in other tables with the aid of this manuscript.

The first 8 figures of the 10,000 12-figure logarithms from 100,000 to 110,000 (all in Volume A_4) were recently compared with the proofs of a table appearing in a new set of tables now in press for Messrs Chambers. The following errors were recorded. Each logarithm in Thomson is written in full, in four triads, and with the characteristic or index.

Number	Triad	*For*	*Read*
100934	1	104	004
101051	2	440	540
101068	2	513	613
101171	Index	4	5
102553	2	958	948
102707	2	599	600
104317	2	455	355
104819	3	112	012
105148	1	121	021
105471	1	123	023
106525	3	342	542
106947	2	178	168
107021	2	468	469
109477	3	287	887

It may not be without interest to record that when the present writer tried in 1926 to see these tables at the library of the Royal Astronomical Society, he was informed that they were out on loan. After persistent efforts by the Secretary, they were received two years later from Dr. Glaisher, who had had them for 47 years! But for this intervention, they might have gone to a bookseller when his library was disposed of after his death in 1928. Now after three quarters of a century, they have once more been useful as an independent check on a published table.

L. J. C.

61[F].—ALBERT GLODEN, *Tables of the Decimal Endings of Cubes, Fourth Powers, and Eighth Powers together with the Linear Forms of the Corresponding Roots.* 6 + 7 + 3 leaves, 21 × 29.7 cm. Typed manuscripts in the possession of the author (rue Jean Jaurès 11, Luxembourg), and of the Brown University Library.

These tables give 1-, 2- and 3-digit endings of cubes, fourth and eighth powers (for the decimal system), arranged in order of magnitude, as well as 4-digit endings of fourth powers. With each such ending are listed all numbers whose corresponding power ends in this way. Thus with the fourth power ending ···3041 the author gives the entry

3041 77,361

This means that all numbers whose fourth power ends in 3041 are given by

$$1250\,n \pm 77, \qquad 1250\,n \pm 361 \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

These tables are extensions to higher powers of corresponding tables, such as those of CUNNINGHAM,[1] concerning squares. The lists of power endings are useful in showing at a glance that a given number is not a power of degree 3, 4, or 8. The rest of the information is useful in setting up exclusion procedures in dealing with diophantine equations of these degrees.

D. H. L.

[1] A. J. C. CUNNINGHAM, *Quadratic and Linear Tables.* London, 1927, p. 89–92.

62[F].—LUIGI POLETTI, *Atlante di centomila numeri primi di ordine quadratico entro cinque miliardi.* Manuscript in possession of the author, Via Cairoli 1, Pontremoli, Italy.

For many years Poletti has been investigating the distribution of primes of the form $ax^2 + bx + c$ for as many as 366 different values of (a, b, c), and now has a list of 116683 primes $> 10^7$ and $\leqslant 5101683361$. The principal forms considered are those for which $(a, b, c) = (1, 1, 1), (1, 1, -1), (2, 2, 1), (2, 2, -1), (1, 1, 17), (6, 6, 31), (3, 3, 1), (3, 3, -1), (1, 21, 1)$ and

$$x^2 + x + 41 \qquad x^2 + x + 72491$$
$$x^2 + x + 19421 \qquad x^2 + x + 146452961.$$
$$x^2 + x + 27941$$

These last five were chosen for study on account of their apparently high density of primes among the numbers they represent. The first of these is due to EULER and admits of no prime factor < 41. The second was suggested by D. H. L. and admits of no factor < 47. The third and fourth are due to N. G. W. H. BEEGER and also admit of no factor < 47. The last is due to D. H. L. and admits of no factor < 109. (See *Sphinx* v. 6, 1936, p. 212–214; v. 7. 1937, p. 40; v. 9, 1939, p. 83–85.) These 5 series have been pushed to high limits x, in fact to $x = 55102, 32147, 16356, 16345$ and 70400 respectively. The corresponding numbers of primes found are 18667, 11473, 6897, 7016 and 27858. The percentages of primes in these five cases are thus .3388, .3569, .4216, .4292 and .3957. It is interesting to note that it has never been proved that any one of these quadratic progressions contains infinitely many primes.

D. H. L.

AUTOMATIC COMPUTING MACHINERY

This new Section will deal with matters pertaining to large-scale automatically-sequenced computing machinery. The wartime need for ultra high-speed calculations has caused a development of the field which may well have a profound effect upon methods of classification and compilation of data and of numerical computation. As a result of this activity, there

now exist in the United States four types of large-scale computing machines: (1) the differential analyzer of the Massachusetts Institute of Technology, (2) an electromechanical computing device (the IBM Sequence Controlled Calculator) of the Harvard Computation Laboratory, (3) relay-type computing machines of the Bell Telephone Laboratories, and (4) an electronic computing machine (ENIAC) of the Ballistic Research Laboratory, Aberdeen Proving Ground, Md.

After the introductory article, the material in this Section falls under the general headings of TECHNICAL DEVELOPMENTS, DISCUSSIONS, BIBLIOGRAPHY, and NEWS. The necessary editing is provided by the staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. CANNON, 418 South Building, National Bureau of Standards, Washington 25, D. C.

THE ZUSE COMPUTER [1]

General Description. The Zuse computer is a relay-type digital machine. At present electro-magnetic telephone relays are used except in the storage unit, where special mechanical relays have been introduced. The machine operates upon numbers in dyadic representation and carries out automatically the ordinary algebraic operations, including division and extraction of square roots. Automatic handling of algebraic sign and of decimal point are provided. Numerical input is limited to the use of a manual keyboard, and output to a visual display panel. An automatically-sequenced program can be introduced by means of punched tape; however it must be followed through in a rigidly prescribed manner.[2] Only directions as part of the program, and not numerical values, can be entered by means of the tape. The machine will handle numbers of five significant figures, within a range of magnitude of $10^{\pm 20}$. Multiplying time is of the order of one second. Projected storage capacity is for 1024 numerical quantities, but only 16 cells have been constructed to date.

The machine is constructed of rather rough homemade components, so that it may be assumed that it occupies more space, and must operate more slowly, than would be necessary for a more carefully engineered model. The manual controls together with the visual display panel are built into a unit of approximately the shape and size of a small upright piano. The relay circuits are contained in four large cases roughly $6' \times 3' \times 1'$. The storage component, with space provided for the projected 1024 cells, is contained in two boxes less than a yard square and 15″ high. Zuse estimates about 1000 cells to the cubic yard. In addition to these parts there are an electric motor for operating the mechanical components, a drum commutator for operating the relays, a tape reader, and a tape punch.

Numerical Input and Output. The machine receives five-digit quantities in decimal notation. These are entered by successively pressing a combination of the 10 digit keys and the key for the decimal point. In addition, the decimal point may be shifted 6, 12, or 18 places to the right or left by pressing one of the keys 10^{+6} or 10^{-6} one, two or three times. The numbers so entered appear immediately in the form of illuminated lamps on the display panel. All numbers are entered thus in semilogarithmic decimal form and are automatically converted into dyadic representation. Twenty-two dyadic places are retained, which corresponds to something over six significant figures in decimal notation.

In operation all quantities are taken in dyadic semilogarithmic form with

algebraic sign. Decimal point position and algebraic sign are handled automatically in all calculations.

Output is obtained by a visual display panel comprising seven columns of ten lamps each for the significant figures, with auxiliary lamps to indicate position of the decimal point, algebraic sign, and other special quantities. Any quantity in the machine may be called for on the display panel at any time, and is automatically converted into decimal notation before presentation.

No means of numerical input or output other than the keyboard and the display panel are provided.

Special Algebraic Devices. A special key is provided for zero, the reason being given that this quantity is unsuited for expression in semilogarithmic form. In addition to the special symbol for zero, a symbol, $\ll$, is provided on the display panel, signifying "of too small a magnitude for the scope of the machine." This would appear, for example, as the result of subtracting any number from itself. It is difficult to see the value of this arrangement, since it was admitted that the symbol "0," for exactly zero, could never be obtained as the result of a computation.

Three expressions for infinity are employed. Of these, $+\infty$ and $-\infty$ indicate a result which is positive or negative but numerically beyond the scope of the machine; the third, $\pm\infty$, denotes a large quantity which is ambiguous as to sign, e.g., the result of dividing 1 by 0 or by $\ll$. An additional symbol, 0/0, signifies an indeterminate result, e.g., the result of dividing one very small quantity by another.

The algebraic combination of these various signs for infinite, infinitesimal, and indeterminate quantities is handled automatically in the same fashion as for ordinary numbers; no special attention is demanded of the operator, nor in preparation of an automatic program. Thus, for example, $(+\infty) - (+\infty)$ automatically yields 0/0, etc.

One key is provided for the minus sign, $-$, and corresponding indicators for $+$ and $-$ on the display panel.

Likewise a key and indicator are provided for the imaginary unit, $i = \sqrt{-1}$. However, operation with imaginary quantities is limited to the two rules that when the square root of a negative number is called for, the symbol i will register, and that when two pure imaginaries are multiplied together a minus sign will register. Beyond this, the machine is not equipped for automatic computation with complex numbers.

Arithmetical Unit. The arithmetical unit contains two special storage cells. When numbers have been read into one or both of these cells, a key may be pressed directing that the two numbers be added, subtracted, multiplied or divided; or, in the case of a single number, that the negative, the square root, the double, the half, the 10 fold, the 10th part of the first of these numbers be produced, or that the number be multiplied by π. Provision is planned for squaring, taking the reciprocal, and the maximum or minimum of two quantities, but these operations are not yet available. The result of an operation is in turn stored in the arithmetical unit. By pressing an appropriate key, the result of the operation can be made to appear upon the display panel as fast as it is computed.

The arithmetical operations are carried out by relay circuits. The conversion from decimal to dyadic representation is carried out by the relays

as are used for addition, together with some auxiliary relays for controlling the operation. Information regarding the actual circuits employed was withheld; however, it was stated that the more complicated operations of division and extraction of square roots are carried through by the procedures used in hand computation.

Storage. The storage is planned to contain 1024 cells, each of which will hold a single numerical quantity with its algebraic sign and decimal position. For this purpose mechanical relays are used. These are of a special design, of which the details were not revealed. One layer of 16 cells is now in operation.

The relays appear to consist of a number of thin strips of metal lying between two plates of glass, not more than a quarter of an inch apart. Strips running in one direction through this layer represent the 16 cells, each containing one numerical quantity, while those in the perpendicular direction represent the individual digits, etc., of each cell.

Motion of the strips is controlled by electromagnetic relays that engage or disengage individual strips with an arm providing mechanical impulses at regular intervals. It would appear that motion of a particular strip representing a chosen cell exposes other moving parts, corresponding to each digit position in the cell, to the motion communicated by the various transverse digit strips. The whole mechanism appeared quite compact and simple; however, detailed examination or description was denied.

There is no doubt that this type of storage is more compact than electromagnetic relays. One layer of 16 cells is roughly two feet square and $\frac{1}{4}''$ thick, to which must be added a few inches for the protruding ends and the operating mechanism. This storage was demonstrated and appeared satisfactory with regard to accuracy and speed of operation; however it was stated that some difficulty had resulted from the necessity of using low-quality metal.

In operation a number may be directed into any cell of the storage from either the input keyboard or the computer. The storage does not operate as an accumulator, and it is not necessary to clear a cell before use. If a second number is directed into a cell in which a number is already stored the first number will be completely lost and replaced by the second. Once a number is entered, it remains in the storage until replaced by another. A number in any cell can be read into the computer, or onto the display panel, at will and as often as is required.

Control Unit. The timing of the basic operations of the machine is controlled by an electric motor, which drives a cylindrical commutator operating the relays and also provides the mechanical drive for the storage relays, thus assuring synchronization. The actual speed of operation was stated to be about half what could ultimately be expected. This delay was attributed to difficulties with the electrical circuits rather than to the electromagnetic or mechanical relays themselves.

All operations and transfers are initiated by pressing a control key, and the completion of an operation is indicated by a special lamp on the display panel. Operating time was stated to be about one multiplication per second. This was confirmed in the demonstration. Extraction of a square root was timed at slightly over 5 seconds. Compared with these operating times, and

with the time required for manual input and control, the time required for transfer from one part of the machine to another is negligible. It was stated that the mechanical relays had about the same speed of operation as the electromagnetic ones.

The timing of the sequence of arithmetical operations is controlled by the operator. When the signal flashes, indicating that one operation is complete, a key must be pressed to initiate the next operation. When the machine operates under automatic control, the completion of one operation causes the tape reader to advance and initiate the next.

Automatic Program. The machine may be operated by a punched film. This is ordinary celluloid moving picture film about an inch wide. An order may be punched on to the film in the form of some combination out of 8 holes, occupying two lines across the film. Numerical values cannot be entered by means of a tape, but only directions to transfer numbers between the storage and the computer, or to perform algebraic operations on the quantities in the computing unit. Before an automatic program can begin, all the initial values must be entered manually by way of the keyboard into the proper cells of the storage. After the calculation is complete, or at any time the operator chooses to intervene, the quantities in the storage may be read from the visual display panel.

The tape is prepared by directing the machine through the course of the calculation in exactly the same way, and possibly at the same time, as for non-automatic operation. The program can be of any length compatible with the storage capacity. An important limitation upon programming is that the machine must adhere to a prescribed linear course of operation. It cannot at any point choose between two subsequent programs on the basis of results already obtained; nor can it be directed to repeat automatically sub-programs within the same total program. A further incidental inconvenience is that even such constants as the number 1 must be entered into the storage along with the initial values, or else obtained as the result of trivial calculations. The speed of the machine for automatic operation is somewhat, but not a great deal, faster than for manual operation.

Present Status of Machine. There exists at present only one model of the Zuse machine. As has been indicated, this is incomplete both in mathematical and electrical design, and with regard to the engineering of the components. The machine is the property of Dipl. Ing. Konrad Zuse and is now housed in the cellar of a farm building in the village of Hopferau near Füssen, in southernmost Germany. Zuse has with him one assistant by the name of Stucken.

Improvements are being undertaken in the electrical circuits, as well as the completion of some of the projected algebraic operations which have not yet been installed, and the construction of more storage cells. This work is being carried out under rather primitive conditions and with inadequate material.

Zuse plans ultimately to convert the machine entirely to mechanical relays. On the basis of his experience with the mechanical relays now in use in the storage, and with earlier entirely mechanical machines, he is convinced that this will offer no serious difficulty once materials become available. It is his hope that after perfecting the design of a compact mechanical

relay machine, he will be able to undertake commercial production in quantity.

So far as can be ascertained, Zuse is carrying out his work independently of any interest or assistance from outside sources.

ROGER C. LYNDON

Office of the Assistant Attaché
for Research, American Embassy,
London, England

[1] This Report was prepared for the Office of Naval Research by Dr. Lyndon of the London branch office. The Zuse computer, invented by Konrad Zuse, is of interest in that it represents the work of German scientists in the field of large-scale computing devices. By 1943 Zuse had established his own firm for the construction of the computers he had invented. Two special-purpose aerodynamic machines developed during the war apparently proved successful. A general-purpose algebraic computer was also completed and had passed its test runs prior to the cessation of hostilities. Although these computers are interesting because of their similarity with wartime developments in the United States, it is clear that they are of limited utility in their present state of development. See also BIBLIOGRAPHY, Z-I, no. 3.

[2] ZUSE states in a letter, received since the completion of this article, that it is now possible to enter constants by means of the punched tape; he also points out that with four tape readers and two tape punches, as called for in the design, the machine will be able to repeat programs and to choose between alternative sub-programs. He states further that when the designed wirings are complete, the machine will be fully equipped to handle problems involving complex quantities.

TECHNICAL DEVELOPMENTS

The Selectron—A Tube for Selective Electrostatic Storage

(See frontispiece plate)

We are engaged at the RCA Laboratories in the development of a storage tube for the inner memory of electronic digital computers. This work is a part of our collaboration with the Institute for Advanced Study in the development of a universal electronic computer. The present note describes briefly the principle of operation of the tube, which is still in its experimental stage. It is a summary of a paper presented at the "Symposium of Large Scale Calculating Machinery" at Harvard University on January 8, 1947; see *MTAC*, v. 2, p. 229–238.

The necessity of an inner memory in electronic digital computers has been realized by all designers. The high computing speed possible with electronic devices becomes useful only when sufficient intermediary results can be memorized rapidly to allow the automatic handling of long sequences of accurate computations which would be impractically lengthy by any other slower means. An ideal inner memory organ for a digital computer should be able to register in as short a writing time as possible any selected one of as many as possible on-off signals and be able to deliver unequivocally the result of this registration after an arbitrarily long or short storing time with the smallest possible delay following the reading call.

The selectron is a vacuum tube designed in an attempt to meet these ideal requirements. In it, the signals are represented by electrostatic charges forcefully stored on small areas of an insulating surface. The tube comprises an electron source which bombards the entire storing surface. The insulator can be a circular cylinder coaxial with a standard thermionic cathode. Between the cathode and the storing surface there are two orthogonal sets of

spaced parallel metallic bars, which form a checkerboard of windows creating the corresponding small storing elements on the insulating surface. In the cylindrical structure one set is formed by rings and the other by straight bars spaced angularly around the cylinder. These bars are insulated from each other, making it possible to apply positive or negative potentials to them and thereby to stop the flow of electrons through all windows except a desired one. Between the selecting structure and the insulating surface is the collector, a grid-like unipotential electrode. The insulating surface is backed by a metal plate called the capacity plate. The operation of the tube consists of assigning selectively an element of surface for each incoming signal, storing the signal-information on that element, and subsequently detecting the stored information identified by its previously assigned location. The selecting and storing mechanisms will now be described separately.

Consider one of the sets of selecting parallel bars. Electrons will pass between two adjacent bars when they are both at the same positive potential with respect to the cathode. On the other hand, if both bars are substantially negative, the electrons will be blocked by a negative potential barrier in front or in the gate formed by wires. When one bar is positive and the adjacent one negative, electrons will also be stopped, provided the geometry of the bars and voltage levels are properly chosen. It is clear, therefore, that if another set of parallel bars is placed at right angles behind the first, electrons will pass through a window limited by two pairs of bars only if all four bars are positive. For a large checkerboard of windows, the number of control voltages and consequently the number of leads to be sealed through the vacuum envelope would be very large if each bar had to be controlled separately. However, this is not necessary, because the fact that a coincidence of both limiting bars is necessary for the opening of a gate makes it possible to connect internally the bars of any one set in groups and control only the potential of a relatively small number of groups. Single positive wires surrounded by negative ones do not open any gates and, therefore, can be connected to the wires of the selected open gate. There are many connection systems solving the combinatorial problem of how to group the elements of each row such that each group contains one element neighboring with an element of each of the other groups, once and once only. As an example, in a system used in some experimental tubes, there are 64 selecting bars in each direction, connected in 16 groups of 4 each. These groups are divided into two families identified by 1, 2, 3, 4, 5, 6, 7, 8 and 1′, 2′, 3′, 4′, 5′, 6′, 7′, 8′. The enumeration of the bars according to the group to which they belong is as follows: 1, 1′, 2, 2′, 3, 1′, 4, 2′, 5, 1′, 6, 2′, 7, 1′, 8, 2′; 1, 3′, 2, 4′, 3, 3′, 4, 4′, 5, 3′, 6, 4′, 7, 3′, 8, 4′; 1, 5′, 2, 6′, 3, 5′, 4, 6′, 5, 5′, 6, 6′, 7, 5′, 8, 6′; 1, 7′, 2, 8′, 3, 7′, 4, 8′, 5, 7′, 6, 8′, 7, 7′, 8, 8′, from which it is apparent that each non-primed group has an element neighboring with an element of every primed group once and once only. In this example, $16 + 16 = 32$ sealed leads control $64 \times 64 = 4096$ elements. More efficient combinatorial systems are possible, particularly with several successive sets of bars in each direction. Anyhow, the number of necessary seals in the indicated system for which N leads control $(\frac{1}{4}N)^4$ elements is relatively so small that it presents no technological limitation even for many elements (e.g., 128 seals can control 1,048,576 elements).

The storage mechanism is based on the fact that an insulating surface exposed to electron bombardment will assume naturally one or the other of two stable equilibrium potentials for which the net electron current will be zero, the cathode potential for which electrons cannot reach the surface for lack of energy or the potential of the collector for which the primary and resulting secondary electron currents are exactly equal. These equilibria are stable because any potential deviation, as could occur from imperfect insulation, for example, results in stabilizing electron currents of a direction proper to bring the surface back to equilibrium. The potential of the collector, the electrode determining the potential gradient at the target surface, may be several hundred volts. It must be sufficiently high for the intrinsic secondary emission ratio to be greater than one. When all the surface of the insulator is bombarded, some elements of surface can be stably maintained at the high collector potential, while others are simultaneously maintained at the low cathode potential. This is an ideal condition for the quiescent state of the memory tube in its stand-by condition. It can be obtained by the simple expedient of making positive all the selecting bars and thereby opening all the windows. The pattern of the equilibrium potentials is "written" into the tube, one element at a time, by closing momentarily all windows except a chosen one and overpowering the electron current locking mechanism remaining on the corresponding element by a displacement current resulting from a voltage pulse applied to the backing capacity plate. The polarity of the pulse is made to depend on the on-off signal assigned to that element and determines to which of the two stable potentials the element will be driven. The "reading," also one element at a time, is obtained by closing momentarily all windows except the one identified by its previously assigned location in the tube and detecting at which of the two potentials the element finds itself. This detection can be by means of a displacement current or with special targets, by a direct electronic current. Another method, convenient for monitoring in any case, consists of coating the insulator with a cathode-luminescent material and making the backing capacity plate semitransparent. Clearly, light will be produced by electron impact for the high but not the low equilibrium potential. This signal can easily be detected and amplified by a multiplier photo-tube viewing the whole storing surface. It is apparent that this method of storage provides for an indefinite storing time, for writing without previous erasing, and repeatable readings.

The tentative engineering characteristics of the selectron tube which we are engaged in developing are: Size from 3 to 4 inches in diameter, 4 to 6 inches long, 50 lead stem, capacity of several thousand elements; and writing and reading times of about 30 microseconds. A greater storage capacity can be compounded by using a number of tubes. It is convenient to use as many selectrons as there are basic binary places in the computer, or in nonbinary machines as many as on-off signal channels, and to connect the selecting control leads in parallel and operate all writing and reading channels simultaneously.

Jan A. Rajchman

RCA Laboratories

Discussions

Should Automatic Computers be Large or Small?

The extremely high speeds of electronic devices are being utilized in machines for the automatic solution of long and complicated numerical problems. This application is of very recent origin, and we are still in process of exploring its possibilities, and of organizing electronic devices into useful tools of applied mathematics.

In planning any large computer of this nature, it must be kept in mind that the automatic computer is a labor saving device. It does nothing that enough human beings with paper and pencil could not do, given sufficient time. Basically, therefore, the design must be determined by economic considerations, and these considerations force the designer to compromise between the speed and utility of the machine, on the one hand, and the cost of construction and maintenance on the other.

In other words, the designer of a new computing machine must decide whether he will build a large machine or a small one, or perhaps a group of small machines. In view of the diversity of uses to which computers will be put, it is unlikely that the optimum size will be the same in all cases. Certain influences operate in general, however, and it is hoped that this note will help the designers of new computing devices to recognize and evaluate the effects of such influences.

In certain parts of the computation process, it is possible to make a fairly even exchange between over-all speed and number of elements. Thus, an electron path can be opened and closed at the rate of 100,000 times per second, to use a conservative figure. We can imagine a computer in which the addition of two complete numbers is performed in one or two intervals of 1/100,000 second each by the use of a large number of parallel paths. In contrast, we can imagine a computer in which a few paths are used over and over to perform the addition. Very roughly, the number of additions per second is proportional to the number of paths operating simultaneously, and hence to the amount of equipment involved.

An automatic computer, however, contains more than the combinational circuits involved in addition and similar operations. One of its major functions is the storage of numerical and control data. In this part of the system, we cannot imagine means for effecting an even exchange of speed and simplicity. The amount of stored information is a function of the problem being solved and is practically independent of the speed with which the storage mechanism operates. Since this is so, it becomes necessary to supply practically as much storage capacity in a slow machine as in a fast one if the same problems are to be treated. This fact, in itself, is a strong argument in favor of large and fast computers. On the other hand, if the calculating mechanism is to have a relatively low over-all speed, then delays in response of the storage mechanism are less important, and slower storage methods can be tolerated.

On the whole, it may be said without too great exaggeration, that the storage device should be chosen or developed to fit the requirements set up for a given computer, and the calculating system then chosen to operate with that particular means of storage.

Another important element of the automatic computer which affects the compromise is the input-output mechanism. In most existing computers, much of this part of the system is mechanical and relatively slow. In large computers, it has been necessary to supply a multiplicity of key boards and typewriters in order to handle the volume of data required by the machine.

The last major element of an automatic computer that has a bearing on the optimum size of the computer is the switching or path-selecting mechanism. In general, the faster the computation and the greater the number of parallel paths, the more complicated is the switching problem. If a small number of calculating elements is used, then the switching becomes relatively simple, in the sense that few switches are required. The number of switches is roughly proportional to the number of parallel data paths and hence to the speed and size of the computer.

Summarizing the equipment requirements for large and small computers, we find that the calculating, switching, and input-output equipments are roughly proportional to the speed of computation. To the extent that such proportionality holds, economic consideration of these elements does not influence the optimum size of computer one way or the other.

The cost of storage, on the other hand, does affect the economy of the computer. It appears that one should use the largest computer compatible with a chosen type of storage unit.

So far, economics of the computers favor the largest possible machine. However, as the machines increase in size other factors make their appearance. Besides the obvious fact that one reaches the limit of profitable work, there is the traffic factor. The larger the machine, the more administrative detail there is in routing problems through it. Priorities must be established; a large staff of operators and maintenance people must be supervised and made to cooperate smoothly and without interference. Problems must be assembled from a large number of sources and the results distributed.

For all these reasons, there are unavoidable delays in handling the flow of data through a large computer.

Another factor that favors smaller units is the increase of maintenance trouble with increasing size of unit. The breakdown of any one part of a machine makes the entire computer unusable until the part has been repaired or replaced. As the number of parts increases, the chance that some part will fail in a given period increases, and hence the proportion of "down" time increases with the size of the machine, other things being equal.

Finally, there are advantages to be gained in close cooperation between the operator of a computer and the person or group with which the problems arise; or better still, the originator of a problem should have a machine available for his own use. Ordinarily, the larger the computer, the more users there must be to support it, economically, and as a general thing automatic computers can be assigned to individuals only if the computer is relatively inexpensive. Obviously there will be exceptions to this general rule.

In conclusion, then, the writer's opinion is that automatic computers should be designed in the smallest units consistent with the problems to be handled. While the larger mechanisms intended for group operation are easier to design economically, the advantages of the small group or one-man users seem to outweigh the purely mechanical considerations. It is recognized, of course, that "large" and "small" are relative terms, and that,

because of the problems to be solved, a single group of users working on an "irreducible" problem may require a machine that is physically very large; but if the problem is reducible to a set of practically independent problems, then a group of smaller machines seems desirable.

G. R. Stibitz

Bibliography, Z-I

1. George R. Stibitz, *Relay Computers*. Prepared for the National Defense Research Committee, February 1945, v, 70 leaves and appendix, 21.7 × 27.8 cm., printed from manuscript by the photo-offset process.

This report, consisting of 9 chapters and an appendix, treats the performance of numerical operations by relay calculating mechanisms.

In Chapter 1 the author states that the purpose of his memorandum is two-fold: to acquaint those who have computing problems with the potentialities and limitations of relay calculators, and to acquaint those who are familiar with relay circuit designs with the special requirements of the relay computer. No attempt is made to cover the details of actual relay circuit design or lay-out. Continuous and digital computers are discussed from the standpoint of accuracy, operator attention required, construction and maintenance, flexibility, and contrast in the design problems of the new types.

Chapter 2, entitled Computational Processes, includes a definition of numerical computing and a classification of computing processes with corresponding requirements on computing systems. The requirement which is of perhaps the most interest to a reader is that a computing machine must be intelligent enough to make what are called decisions. It is frequently necessary for computing machines to determine from the basis of the results of a computation step which of alternative steps shall be followed. Such decisions are necessary, for instance, in certain step-by-step numerical integrations where successive approximations are computed until an error term has been reduced to a predetermined reliable value.

Chapter 3, entitled Relay Computing Elements, treats the relay as a computing element. The relay is essentially an electrically operated switch. Relays of the type used in the computing mechanism described in the report have two and only two stable positions—the off position and the on position. If the off position is represented by the symbol 0 and the on position by the symbol 1 and if numbers are represented by means of the symbols 0 and 1, the positions of relays in a relay network can be used to represent them. The manipulation of numbers in relay computers is discussed; the use of relays to store numbers and to perform the operations of addition, multiplication, division and square rooting is treated in a general manner.

Chapter 4, entitled Theory of Automatic Checks, contains a discussion of the reliability of relay computing elements and an analysis of "troubles" (a trouble being defined to be a condition not considered in the circuit design) and checking circuits as aids to maintenance. Trouble conditions pertaining to relay computers may be grouped as troubles involving leads and troubles involving contacts. Trouble conditions occur when there is either abnormally high resistance or abnormally low resistance between different points in the system. On the one hand, there might be abnormal conditions existing between two leads or between a lead and some other point in the system. On the other, an abnormally low resistance might occur across relay contacts at a time when the contacts should be open, or abnormally high resistance at a time when the contacts should be closed. The latter condition is much more likely to occur than the former because of the presence of dirt between the surfaces and the contact. Practicable checking circuits are discussed including their use as aids to maintenance of the computing machine.

Chapter 5, entitled The Design Problem-Functional Design Step, consists of a functional analysis of computing machine systems. In the author's words, "it has been found

convenient in designing computing devices for complicated operations to follow a general pattern reminiscent of the human nervous system in the sense that the calculator is arranged to have a number of levels of nerve centers with differing orders of intelligence, ranging from a kind of simple reflex at the bottom of the scale up to an over-all control which integrates and guides the entire operation." A specific computing device, namely a relay interpolator, is used as an example in this Chapter. This interpolator is designed to accept the values of an arbitrary function for integral values of the argument, to compute interpolated values of the function using an interpolating polynomial of degree 3 and to punch the function values in telegraph tape in coded form. The control levels required in such a computing device are listed and a functional schematic is drawn for it.

Chapter 6 is entitled The Design Problem—Translator Design Step. By translator design step is meant the translation of a functional schematic, or block diagram, into the more detailed relay circuitry. An interesting and instructive systematic approach to this problem attributed to Dr. C. E. Shannon is discussed. This approach consists in the application of Boolean algebra to the design of relay circuits to possess given functional ability. The elements of this algebra are given and the conversion of the algebra into relay circuits is treated.

Chapter 7, entitled Radix Notation, gives representations of numbers that are convenient when relays are used. It is pointed out that binary representation of numbers unfortunately does not convert easily to and from a decimal notation and that consequently its use in relay machines is limited. A hybrid representation, in which the bases 2 and 5 are used and which is called the bi-quinary notation by the author, is adopted by him for use in relay computers. This scheme of representation consists of using the base 2 to tell if the decimal digit lies between 0 and 4 or between 5 and 9 and in using the base 5 then to tell which digit it is. The elementary arithmetical operations upon numbers expressed in the bi-quinary notation are discussed very completely in the remainder of the chapter.

Chapter 8, entitled Schematic Diagrams, consists of a discussion of physical and functional schematics and gives some conventions used in such schematics. As illustrations four figures are given. Some advantages and disadvantages of the conventions used are listed.

Chapter 9, entitled Relay Complexes, consists of further discussion of the computer nerve centers discussed in Chapter 5. A relay complex is defined to be an intermediary nerve center controlling a frequently used operation. Relay complexes capable of performing the operations of addition, subtraction and multiplication are discussed. Alternative types of complexes for performing the same function are compared. The details of a nonchecking binary relay adder are given and adders in other bases are discussed. It is pointed out that the combination of a binary adder with a quinary adder constitutes a decimal adder. This is the combination used in the relay interpolator mentioned in the preceding chapter. Relay complexes for storing information on tape and for searching through the stored data for particular desired numbers are discussed.

The appendix is particularly interesting. It is stated that the relay interpolator which was designed to calculate automatically by a cubic formula a set of interpolated values between given data points will solve an astonishing variety of problems. These problems include interpolation, smoothing of data, integration, differentiation, solution of linear differential equations of the first order, solution of linear differential equation with constant coefficients of higher order, harmonic analysis, and the extraction of roots of polynomials. The appendix purports to be an investigation of the underlying principles that lead to so diversified a set of results. This mathematical treatment of the function of the interpolator is one of the most interesting portions of the report.

The flavor of this report by George R. Stibitz has undoubtedly not been retained in this review. As a treatment of underlying mathematical theory of numerical computation by relays, of the general principles of the design of relay networks for computing purposes, and of a discussion of reliability of computing machines from the engineering viewpoint, the report is among the most instructive and interesting that have been written on numerical computers.

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2. National Defense Research Committee, *Description of the ENIAC and Comments on Electronic Digital Computing Machines*, prepared by J. P. Eckert, Jr., J. W. Mauchly, H. H. Goldstine and J. G. Brainerd of the Moore School of Electrical Engineering, University of Pennsylvania. Report dated November 1945. Printed from manuscript, viii, 58 leaves and 11 drawings, 21.7 × 27.8 cm., by the photo-offset process. See also *MTAC*, v. 2, p. 97–110.

This report, consisting of six chapters and a sizable appendix in three parts, gives a description of the ENIAC, the first general-purpose automatically-sequenced electronic digital computing machine.

As stated in the Introduction, the speed of the ENIAC (Electronic Numerical Integrator and Calculator) greatly exceeds that of any nonelectronic machine, and its accuracy is in general superior to that of any nondigital machine, such as a differential analyzer. The ENIAC is extremely flexible and is not fundamentally restricted to any given class of problems. However, there are problems for which its speed is limited by the input and output devices. In such cases it is not possible to derive the full benefit of its high computing speed. Although the ENIAC carries out its entire computing schedule automatically, the sequence to be followed must be set up manually beforehand. However, as the intended use of the ENIAC is to compute large families of solutions all based on the same program of operations, the time spent in setting up the problem can usually be disregarded.

A second electronic digital machine, the EDVAC, is now being planned. Completely automatic, it will be of larger capacity than the ENIAC and will have a somewhat higher computing speed. Despite these features, it will require considerably less equipment than the ENIAC, since the electronic components will be used in a quite different and much more efficient way.

Chapter 1 gives an interesting account of some of the computing problems that led to the organization of a project for the design and construction of the ENIAC. The most pressing need at that time was for a computing device capable of making firing tables quickly to keep pace with the development of gun-projectile propellent combinations. The nature of the problems which these machines bring within the range of computation and the real need for speed in carrying out such calculations are discussed.

Also included in Chapter 1 is a brief history of the growth of the ENIAC project. Early in the spring of 1943, Captain Herman H. Goldstine of the Ballistic Research Laboratory and Colonel Paul N. Gillon of the Office of the Chief of Ordnance became interested in the utilization of an electronic computer for the preparation of firing and bombing tables. At Goldstine's request, J. W. Mauchly and J. P. Eckert, Jr., wrote a tentative technical outline of a machine capable of numerical integration of trajectories and of handling other problems of similar complexity. This material was included by J. G. Brainerd in a report that formed the basis of a contract between the University of Pennsylvania and the Government to develop an electronic device along these lines. Eckert was chief engineer on the project, Mauchly acted as principal consultant, and Goldstine was appointed resident technical representative of the Ordnance Department.

In Chapter 2 the point is made that, in order to attain the speed, accuracy, and flexibility required for the solution of problems such as those discussed in Chapter 1, electronic digital machines must be used.

A description of the ENIAC in Chapter 3 gives the reader a rather complete account of its general features and also provides him with some information on the way in which the various units can be used. The important function of control, various kinds of memory or storage facilities, the arithmetic units, and the input and output devices are considered. (Specific explanations of some arithmetic and programming techniques will be found in the appendices; there is also an appendix giving constructional data.)

The latter chapters of the report are concerned with a discussion of some general prin-

ciples which seem pertinent to computing machine design and which have been used in formulating the plans for the EDVAC.

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3. K. Zuse, *Calculator for Technical and Scientific Calculations Designed According to a Theoretical Plan*. 59 p. 21.6 × 28 cm. Some pages will not reproduce well. Distributed by the Office of the Publication Board, Department of Commerce, Washington 25, D. C. $4.00 (photostat), $2.00 (microfilm).

This report is interesting as an indication of the recent thinking in Germany concerning automatically-sequenced computing machines. It consists of an introduction; brief sections on algebraic calculating devices, logical calculating devices, and the construction of computing devices; and three rather verbose appendices amplifying each of the sections.

In the Introduction the author states that his report is concerned with machines capable of logical inference. Calculating is defined as the deduction of conclusions from given assertions by the application of a prescribed set of rules. He states that calculating thus defined is based on the fundamental logical concepts of conjunction (A and B), disjunction (A or B), and negation (not A) and that it can be expressed throughout, from assertion to conclusion, in a language of "yes-no" values. Numerical computation then becomes a special case of calculating as defined. Calculating machine elements therefore need only be capable of distinguishing between two contrasting states—for example, two states of voltage, of current, or of physical position. It follows that electrical or mechanical relays are inherently suitable for use as the basic elements in calculating machines.

Algebraic and logical calculating devices are also discussed in the Introduction. It is apparent that the two are one and the same, namely, a simple relay computing machine, capable of performing as high as fifty basic arithmetic operations a minute. The binary representation of numbers constitutes the "yes-no" series. It is claimed that the machine can convert automatically decimal numbers. The device can be sequenced by hand manipulation of a keyboard or automatically by perforated tape, and can store both original data or intermediate results internally, operating at the appropriate times upon the stored numbers. The machine can execute automatically order sequences of any length, but it is incapable of modifying orders, that is, of selecting an order routine according to the results of previous computations. Zuse states, however, that the method of constructing devices capable of deducing the order sequences required for the solution of given problems is clear. He foresees the use of machines of this nature to do elementary thinking and thus free man for deeper reasoning and philosophical pursuits.

Section 1 and Appendix 1 list the operations considered basic for an algebraic calculating machine and present the order sequences for the solution of various simple problems. Appendix 2 treats a more complicated problem—the determination of the zeros of a third order determinant having complex elements that are functions of two variables. This determinant occurs in the flutter analysis of plane airfoils with three degrees of freedom.

Zuse calls fundamental the operations of addition, subtraction, multiplication, division, and extraction of the square root. He mentions additional operations that a calculating machine should be capable of performing, ranging from extracting the sign of a number to selecting the smaller or larger of two unequal numbers. He states that the performance of the algebraic calculating machine can be extended to include operations on complex numbers, the use of trigonometric and hyperbolic functions, the determination of the roots of algebraic equations of the third and higher degrees, interpolation, and the numerical integration of differential equations.

Section 3 and appendix 3 enumerate applications of a logical calculating machine. These include the derivation of all mathematical theorems that follow from a given set of axioms, the performance of combinatorial reasoning, exercises in Boolean logic, the preparation of order sequences for computing machines, the simplification of literal algebraic expressions, and studies in aerodynamics, heat flow, optics and probability.

Zuse's report is rather general. More detailed information on the actual construction and use of the machines discussed would have been of interest to many readers. For example, the coding of the numbers and the order sequences as well as the speeds required for the performance of the fundamental operations of addition, multiplication, division, and extraction of the square root might well have been included. Also in reading the report one becomes curious about the types of relays used as basic computing elements.

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4. HARVARD UNIVERSITY, Computation Laboratory, *Annals* v. 1: *A Manual of Operation for the Automatic Sequence Controlled Calculator*. Cambridge, Mass., Harvard University Press, 1940, xiv, 561 p. and 9 plates. 20 × 26.7 cm. $10.00.

This volume was reviewed by Professor D. H. LEHMER in *MTAC*, v. 2, p. 185–187. The review states: "This volume gives the first really scientific account of the Automatic Sequence Controlled Calculator, the first of the large all-purpose digital calculators developed during the war." The reviewer gives an accurate and comprehensive account of the contents of this manual of operation for the large-scale electro-mechanical computing machine developed jointly by Harvard University and the International Business Machines Corporation.

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NEWS

American Institute of Electrical Engineers.—At the Winter Meeting held in New York City from January 20 to 31, 1947, the following series of talks on electronic digital computing machines were given:

JOHN VON NEUMANN of The Institute for Advanced Studies pointed out the inherent error caused by (1) random noise in the analogy machine and (2) the inability of digital computers to carry numbers to their full places. He emphasized that the digital technique is intrinsically the better method of keeping the error small. The two greatest difficulties of the machine, said Dr. von Neumann, are those of programming problems for the machine and of getting data from physical measurements on which the machine can operate to produce meaningful answers.

The use at Harvard University of the IBM automatic sequence-controlled calculator was described by H. H. AIKEN, who likewise called attention to the problem of the prearranged controlled sequences that must be worked out by trained mathematicians before the machine can proceed. Once these program orders have been set down for a given type of problem, many similar problems can be solved merely by substituting the appropriate numerical values in the original set of program orders.

JULIAN BIGELOW of Princeton, in a discussion of the thinking that goes into the design of a computer, stated that, far from the popular misconception of the solution on these machines of problems heretofore unsolvable, mathematicians must think through beforehand all manipulations which the computer executes for an ultimate solution. Hence, the first consideration in making efficient use of computers is to simplify the programming. Mr. Bigelow called attention specifically to the external magnetic-tape memory, permitting storage of a great many numbers in a very small space. This was contrasted with the internal electronic memory, which has the desirable feature of being instantly readable at any point.

An all-electronic computer that can be adapted to the specific problems of industry as well as those of pure and applied science, engineering, and statistical studies was described by J. W. FORRESTER of the Massachusetts Institute of Technology. New features include the transmission of 40 digits of a number simultaneously over parallel busses (introducing the circuit problem of precisely gating each train of pulses to the proper unit of the computer), electronic storage on a dielectric plot, and the possibility of manual programming for locating faults in the machine.

T. K. Sharpless of the University of Pennsylvania described the ENIAC—a machine developed for the specific purpose of preparing ballistic firing tables; see *MTAC*, v. 2, p. 97–110. He also mentioned the general purpose computer EDVAC, which has greater memory facilities, although fewer tubes. The EDVAC, however, does not perform operations quite so fast, because it uses time sequence rather than the spatial division of the ENIAC. The coding is done by magnetic tape prepared at slow speed and run through the calculator at high speed.

A relay-operated computing device produced by Bell Telephone Laboratories was described and illustrated by S. B. Williams, consulting engineer. Controlled by a punched tape, the machine adds, subtracts, multiplies, divides, extracts square roots of real or complex numbers, and prints results on teletypewriters. If the computation called for by the control tape is not completed in the allotted time, the machine stops and signals a fault.

The problem of accuracy of calculations performed by automatic computers was outlined by John Mauchly of the Electronic Control Company. Inaccuracies produced by rounding off in digital machines introduces errors whose magnitudes are difficult to estimate. The machines themselves can be used to determine their truncation errors by solving the problem in several ways.

The Actuarial Society of America (393 7th Avenue, New York).—The Council at its meeting in February authorized the President, E. W. Marshall, Vice President, Providence Mutual Life Insurance Company, Philadelphia, to appoint a committee to explore the possibilities of electronic sequence-controlled calculators. The committee appointed consists of:

Malvin E. Davis (Committee Chairman), Actuary, Metropolitan Life Insurance Company, 1 Madison Avenue, New York;

Pearce Shepherd, Vice President and Associate Actuary, Prudential Insurance Company of America, Newark, N. J.; and

William P. Barber, Jr., Secretary, Connecticut Mutual Life Insurance Company, Hartford, Connecticut.

This Committee has had several meetings. On June 25 it met at the Harvard Computation Laboratory in Cambridge.

At the meeting of the Society at the Hotel Commodore in New York, on May 8, 1947, Edmund C. Berkeley presented a paper entitled "Electronic machinery for handling information, and its uses in insurance."

The Life Office Management Association.—In March, the President, Horace W. Foskett, Vice President of the Equitable Life Insurance Company of Des Moines, Iowa, appointed a committee of the Life Office Management Association on electronic sequence-controlled calculators. This committee consists of:

Charles H. Yardley (Committee Chairman), 2nd Vice President and Comptroller, Penn Mutual Life Insurance Company, Philadelphia, Pa.;

Lester H. Van Ness, Supervisor, Planning Division, Acadia Mutual Life Insurance Company, Washington, D. C.;

Burgh S. Johnson, Treasurer, Guardian Life Insurance Company, New York;

R. A. Mangini, Manager, Planning Division, John Hancock Mutual Life Insurance Company, Boston, Mass.;

Frank L. Rowland, Executive Secretary, Life Office Management Association (ex officio);

H. W. Foskett, Vice President, Equitable Life Insurance Company, Des Moines, Iowa (ex officio);

E. C. Berkeley (Committee Secretary), Chief Research Consultant, Prudential Insurance Company of America, Newark, N. J.

The first meeting of this committee was on May 9, 1947. The second meeting was on June 25 at the Harvard Computation Laboratory, at the invitation of Professor Howard H. Aiken.

Institute of Radio Engineers.—Tuesday afternoon, March 4, 1947, was devoted to

discussions of Electronic Digital Computers at the 1947 National Convention of the IRE. The meeting was presided over by HARRY DIAMOND of the National Bureau of Standards, Washington, D. C. Speakers were as follows:

J. W. FORRESTER of MIT discussed "The electronic digital computer" (with mention of early attempts and existing systems), a general block diagram of a modern proposed computer, and an outline of the fundamental computer operations.

S. N. ALEXANDER of the National Bureau of Standards in his talk on "Input mechanisms for electronic digital computers" established criteria for acceptable input mechanisms. He also discussed recently developed input systems and special materials used in them.

H. H. GOLDSTINE of the Institute for Advanced Study, Princeton, New Jersey next talked on "Electronic computing" and demonstrated how arithmetical operations as well as switching of numbers and control of computation can be realized by means of vacuum-tube circuits.

"The Selectron—A tube for selective electrostatic storage" was described by J. A. Rajchman, RCA Laboratories Division, Princeton, New Jersey. See under TECHNICAL DEVELOPMENTS of this issue.

P. CRAWFORD, Special Devices Division, Office of Naval Research, Washington, D. C., terminated the discussions with a talk on "Applications of electronic digital computers," which included comments on the future relation of analogue and digital computers, and also on the possible engineering application of electronic digital computers to automatic process and factory control, traffic control, and business calculations.

Because of the great interest in the computer program these talks were repeated the same day from 5 to 7 p.m.

OTHER AIDS TO COMPUTATION

See also our introductory article "Film Slide Rule," and QR 30.

BIBLIOGRAPHY, Z-I

1. CARLTON E. BROWN. "The use of double-cycle A and B scales on straight slide rules," *Science*, v. 101, May 18, 1945, p. 522.

2. RAYMOND DUDIN, *La Règle à Calcul*. Paris, Dunod, 1945, xii, 139 p. 12 × 15.8 cm.

3. JOSEPH T. HOGAN, "Slide rules for rapid solution of special equations," *The Chemist Analyst*, Phillipsburg, N. J., v. 34, 1945, p. 29–39. Explanations as to how to make slide rules for special purposes.

4. ANDREW R. WEBER, "Slide rule applications to algebraic equations—solution of the cubic equation," *Jn. Engin. Educ.*, v. 35, 1945, p. 507–514. See also Weber's "Slide rule applications to algebraic equations" [quadratic]. *idem*, v. 33, 1943, p. 775–780.

5. STEFAN BERGMAN, "Punch-card machine methods applied to the solution of the torsion problem," *Quart. Appl. Math.*, v. 4, Apr. 1947, p. 69–81. "The present paper illustrates the application of orthogonal functions to the solution of Laplace's equation $(\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial y^2) = 0$ through the use of punch-card machines."

6. E. G. COX, L. GROSS & G. A. JEFFREY, "Use of punched card tabulating machines for crystallographic Fourier syntheses," *Nature*, v. 159, 29 Mar. 1947, p. 433–434.

7. JACK LADERMAN & MILTON ABRAMOWITZ, "Application of machines to differencing of tables," Amer. Statistical Assoc., *Jn.*, v. 41, June 1946, p. 233–237.

Last paragraph: "The foregoing Underwood-Elliott Fisher Accounting machine [Model D], a standard business machine, accomplishes more than Babbage expected of his Difference Engine and represents a marked improvement over other machines used to difference tables. The ease with which the operations of the machine can be changed, its comparative simplicity, easy manipulation, and low cost, greatly enhances [*sic*] its practicality and demonstrates [*sic*] how a machine of this type lends itself to specialized computational work. Unquestionably, many more machines of this general nature exist which appear to be unknown to scientific computers. There is no doubt that the great effort currently expended on computational work could be appreciably reduced by stimulating a wider knowledge of the capacities of existing business machines and by promoting a broader extension of their applications."

8. O. AMBLE, "On a principle of connexion of Bush integrators," *Jn. Sci. Instruments*, v. 23, 1946, p. 284–287. See *Math. Rev.*, v. 8, 1947, p. 288.

9. G. HÄGG & T. LAURENT, "A machine for the summation of Fourier series," *Jn. Sci. Instruments*, v. 23, 1946, p. 155–158. See *Math. Rev.*, v. 8, 1947, p. 56.

NOTES

75. BHOLANATH PAL'S TABLES OF ROOTS OF THE EQUATIONS $P_n^m(x) = 0$ AND $dP_n^m(x)/dx = 0$ REGARDED AS EQUATIONS IN n.—These tables are given in Calcutta Math. Soc., *Bull.*, v. 9, no. 2, 1919, p. 95, and v. 10, 1919, p. 188–194. The tables contain 4S values of n, for each of 87 zeros of the equations, $\theta = 15°(15°)45°$, $x = \cos\theta$, except for 9 3S values. The tables give also numerous numerical details of the calculations. In Amer. Math. Soc. *Bull.*, v. 53, Feb. 1947, p. 154–155 it was stated by C. W. HORTON that Pal erred in listing for $P_n^2(x) = 0$ the roots $n = 4.77, 2.26, 1.52$, corresponding respectively to the values $\theta = 15°, 30°, 45°$. Horton also supplied 9 other early zeros which Pal had overlooked. These are for $P_n^0(x) = 0$ and $dP_n^1(x)/dx = 0$, $\theta = 15°(15°)45°$, and for $dP_n^2(x)/dx = 0$, $\theta = 15°, 30°$. These 9 values are given by Horton in tables including exact reprints of Pal's 87 values of the zeros.

Pal points out that the roots of these equations are of very great importance in a number of physical problems involving a conical boundary. In illustration Pal gives references to papers by CARSLAW[1] dealing with scattering of sound waves, and to the discussion by LAMB[2] of the problem of the determination of the oscillations of a sea bounded by parallels of latitude. In his computations Pal used an asymptotic expansion due to G. N. WATSON (Camb. Phil. Soc., *Trans.*, v. 22, 1918, p. 277–308).

R. C. A.

[1] H. S. CARSLAW, *Math. Annalen*, v. 75, 1914, p. 133f., 592; *Phil. Mag.*, s. 6, v. 20, 1910, p. 690–691.

[2] H. LAMB, *Hydrodynamics*, third ed., Cambridge, 1906, §200, p. 292; German transl., Leipzig and Berlin, 1907, p. 359f; and sixth ed., 1932, §201, p. 306.

76. MARTIN WIBERG, HIS TABLES AND DIFFERENCE ENGINE.—Brown University has recently acquired a copy of Wiberg's large volume, *Tables de Logarithmes Calculées et Imprimées au moyen de la Machine à Calculer*, Stockholm, Compagnie anonyme de Forsete, 1876. xii, 561 p. 17 × 27 cm. The text and table headings are entirely in French, and the "Avertissement," p. iii–iv, by the author is dated "Stockholm, en novembre 1875." "1875" was the date of a Swedish edition entitled *Logarithmtabeller uträknade*

och tryckte med Räknemaskin (of which there is a copy in the Boston Public Library), and there were also English and German (1876, *Fortschritte*) editions.

There are the following six 7-place tables in the volume: T. I, p. 1–185, log N, $N = 1(1)100000$, with P.P. and values of S and T for $0(50'')1000''$-$(10'')9990''$, i.e. to $2°\,46'\,30''$. T. II, p. 186–189, ln N, $N = 1(1)1000$; T. III–IV, p. 190, Multiples, 0(1)99, of ln M, and of $1/M$; T. V, p. 191–291, log sin x and log tan x for $x = 0(1'')5°$; T. VI, p. 292–561, log sin x, log tan x, log cot x, log cos x, for $x = 0(10'')90°$, with Δ and P.P.

The title of Wiberg's volume suggests that the tables were calculated anew, but concerning such calculation there is not one word in the text. The last sentence of the "Avertissement," is "La disposition des tables est principalement en conformité de celles de Bremiker et de Dupuis." The tables thus referred to are, presumably, JEAN DUPUIS, *Tables de Logarithmes à sept Décimales d'après Callet, Véga, Bremiker, etc.*, Paris, 1862, and the 7-place table, VEGA, *Logarithmisch-Trigonometrisches Handbuch*, edited by BREMIKER, Berlin 1856. [Bremiker's own table, first published in 1852, was a 6-place table.] The "disposition" of each of these tables is in broad outline the same as that of Wiberg.

The original of the machine by means of which the tables were "calculated and printed" was examined by a commission, appointed by the French Academy of Sciences, and consisting of MATHIEU, CHASLES & DELAUNAY. Their report to the Academy is published in Acad. d. Sci., Paris, *C.R.*, v. 56, 1863, p. 330–339; this suggests how the tables may have been calculated. It is noted that already in 1863 interest tables had been calculated and published by the aid of the machine. The report concludes: "Nous proposons à l'Académie d'accorder son approbation à cette belle et ingénieuse machine." There is a brief statement about the machine in M. D'OCAGNE, *Le Calcul Simplifié*, third ed., Paris, 1928, p. 77–78. This second Swedish Difference Engine performed exactly the same things as that of the SCHEUTZS, but by the introduction of new mechanical devices it occupied very much less space. A picture of the machine is given on the outside paper cover of this volume of tables. A copy of an edition of the Wiberg interest tables referred to above is in the British Museum, and has the following title: *Med Maskin uträknade och stereotyperade Ränte-tabeller, jemte en Dagräknings-Tabell* . . . Second enlarged ed., Stockholm, 1860.

The printing of tables by the Scheutz machines (in the publications we have previously listed, *MTAC*, v. 2, p. 242–243) was exceedingly unattractive. Wiberg tells us that the delay of more than a decade in publishing a volume of tables, was largely due to long research for achieving typographic excellence. The result, involving an attractive face of type, with top and bottom tails, and of different sizes, is certainly a great advance over the product of the Scheutz machines, some 20 years earlier.

The Wiberg Tables (1876) do not seem to be very generally known; they are not mentioned in J. HENDERSON, *Bibliotheca Tabularum Mathematicarum*, 1926, or in the printed catalogues of Library of Congress, British Museum, Astor Library, Univ. of Edinburgh, Royal Observatory of Edinburgh, Hamburg Math. Soc., Frankfurt City Library, Amer. Math. Soc., and they were not seen by FMR. These Tables are not in the University Libraries of Columbia, Cornell, Harvard, Illinois, Michigan, or in the New York Public

Library or Library of Mass. Institute of Techn. Those two copies of Wiberg's Tables, to which we have referred, seem to be the only ones in this country at present. Does any reader know that the accuracy of the Tables has been investigated?

Wiberg was an exhibitor at the Philadelphia Exhibition of 1876. In U. S. CENTENNIAL COMM. *International Exhibition 1876. Reports and Awards*, ed. by F. A. WALKER, v. 7, Washington, 1880, p. 165, is the following under Wiberg's name and address: "Exhibits logarithmic tables calculated and printed by machine of his invention, by which means the results are free from errors; also a deep-sea drag, of excellent design, for exploration of the sea bottom."

Wiberg was born in 1826, got his doctorate at the University of Lund in 1850, and died at Stockholm in 1905. According to "Poggendorff," v. 3_2, 1898, among his inventions was an apparatus for railway train heating, and a machine for measuring the speed of a train. There is a portrait and sketch of Wiberg in HERMAN HOFBERG, *Svensk Biografiskt Handlexicon*, new ed., Stockholm, 1906, v. 2.

R. C. A.

77. II.—The famous American astronomer and mathematician, SIMON NEWCOMB (1835–1909) once remarked concerning the calculation of π: "Ten decimal places are sufficient to give the circumference of the earth to the fraction of an inch, and thirty decimals would give the circumference of the whole visible universe to a quantity imperceptible with the most powerful telescope." E. KASNER & J. NEWMAN, *Mathematics and the Imagination*, New York, 1940, p. 78.

78. SEISMOLOGICAL TABLES INVOLVING SIMPLE MATHEMATICAL FUNCTIONS.—The tables about to be analyzed are in *Prince* BORIS BORISOVICH GALITZIN, *Seismometrische Tabellen . . . Nachtrag zu der Abhandlung "Ueber ein neues aperiodisches Horizontalpendul mit galvanometrischer Fernregistrierung," Comptes Rendus des séances de la Commission Seismique Permanente*, v. 4, part 1, St. Petersburg, 1911. ii, 266 p. The volume contains 17 tables all of which may be described as if they were simple mathematical functions. We indicate the contents of Tables **1–9**.

T. 1, p. 19–21, $(1 - x^2)^{\frac{1}{2}}$, and $e^{\pi t}$, where $t = (1 + x^2)^{\frac{1}{2}}/x$, for $x^2 = [.01(.01)1; 2\text{–}3S]$.
T. 2, p. 23–64, T_p/T, for $T_p = 1(1)40$, $T = [10.1(.1)30; 3D]$, Δ.
T. 3, p. 65–71, $\log(1 + x^2)$, for $x = [.01(.01)4; 4D]$, Δ.
T. 4, p. 73–82, $\log[2x/(1 + x^2)]^2$, for $x = [.01(.01)4; 4D]$, Δ.
T. 5, p. 83–207, $\log U$, $U = (1 + x^2)[1 - 2xm^2/(1 + x^2)]^{\frac{1}{2}}$ for $x = [.01(.01)2; 4D]$, Δ, and $m^2 = -.1(.01) + .2, .6(.01).9$.
T. 6, p. 209–213, $(2\pi)^{-1}\tan^{-1}[2hx/(x^2 - 1)]$, $h = (1 - m^2)^{\frac{1}{2}}$, for $x = [.1(.1)4; 3D]$, Δ, and $m^2 = -.2(.1) + .9$.
T. 7, p. 215–217, $(2\pi)^{-1}\tan^{-1}[2x/(x^2 - 1)] + \frac{1}{4}$, for $x = [.1(.1)4; 3D]$, Δ.
T. 8, p. 219–222, $\Delta m = \frac{1}{3}m^3/D^2 - \frac{1}{5}m^5/D^4 + \frac{1}{7}m^7/D^6$, $D = 10^3$, $m = [51(1)400; 1D]$.
T. 9, p. 223–232, $\log(1 + .5372x^2)^{\frac{1}{2}}$, for $x = [.001(.001).8; 5D]$, Δ.

R. C. A.

79. TABLES OF 2^n.—A table for $n = 1(1)120$ is given by PETERS & STEIN, *Zehnstellige Logarithmentafel*, v. 1, *Anhang*, Berlin, 1922, p. 13–32; for $n = 1(1)140$ by H. W. WEIGEL, *$x^n + y^n = z^n$? Die elementare Lösung des Fermat-Problems.* Leipzig, 1933; for $n = 1(1)50$, $52(4)100(5)150(6)180$ by BAASMTC, *Table of Powers.* Cambridge, 1940; for $n = 1(1)256$ by ALEKSANDER KATZ, *Riveon Lematematika*, v. 1, April, 1947, p. 83–85; for $n = 1(12)721$ by WILLIAM SHANKS, *Contributions to Mathematics Comprising chiefly the Rectification of the Circle.* London, 1853, p. 90–95. In *MTAC*, v. 2, p. 246, references were made to manuscript tables, for $n = 1(2)1207$ by Dr. J. W. WRENCH, JR., and for non-consecutive values of n up to 671 by Professor H. S. UHLER. In *Intermédiaire des Recherches Mathématiques*, v. 2, July 1946, p. 73, no. 0550, what purports to be the value of 2^{600} is given by D. FAU who asks if higher powers of 2 have been calculated. The seventy-sixth digit in the 181-digit value given should be 2, not 8, and the hundred and fifty-fifth and fifty-sixth digits 86, should be 68. Professor Uhler's later calculation has been to $n = 1000$, and checked as agreeing with the value obtained by Dr. Wrench in each of its 302 digits.

R. C. A.

QUERIES

23. HENDRIK ANJEMA.—What is known besides facts indicated below concerning this author of *Table of Divisors of all the Natural Numbers from 1 to 10000*, Leyden, 1767 (of which there were also Dutch, French, German and Latin editions of the same date)? In this volume's "Avertisement of the Booksellors" are the following notes regarding Anjema, "After having taught with success & applause, for several years, the Mathematicks in the University of Franequer, he resolved to devote the leisure hours, which an Employment given by the states of Friesland had left him, to the advantage of his old Disciples. He formed the design, of giving a Table of Divisors of all the natural numbers to the amount of 100 000, & he had already brought it so far as 10000, when unhappily he died." What was the year of death of the author of this posthumously published work? A. v. BRAUNMÜHL, in M. CANTOR, *Vorlesungen über Geschichte der Mathematik*, v. 4, 1908, misspells the name as "Ajema" (p. 434, 1099).

R. C. A.

QUERIES—REPLIES

30. LOG LOG TABLES (**Q4**, v. 1, p. 131; **QR9**, p. 336, **12**, p. 373).—In MARCEL BOLL, *Tables Numériques Universelles*, Paris, 1947, p. 446–449, there is a 5D table of log log x, for $x = 1.001(.001)1.18(.002)1.3(.005)1.5(.01)2(.02)3(.05)5(.1)10(.5)20(1)50(2)100(5)200(10)300(20)500(50)1000(100)2000(200)5000(500)10000(1000)20000(5000)50000(10000)100000$.

In *Electrical World*, v. 122, Oct. 14, 1944, p. 118–119, is an article by JERRY AILINGER, "24-scale slide rule solves vector problems." This "Rota-Vec-Trig" rotating slide rule invented and designed by the author is designated as a log log vector and trigonometric slide rule. For greater flexibility there are log log scales in the hexagonal sliding section. There are also new log log scales of decimal quantities of full unit length for greater accuracy,

referred to the D scale, which give the values of e^{-x} with a single setting." See also *Electronics*, v. 17, Sept. 1944, p. 252.

D. S. Davis, "Reading friction factors from a log-log slide rule," *Chem. and Metallurgical Engineering*, v. 51, July, 1944, p. 115. A table shows the very close correlation of results obtained both graphically and by this slide rule.

R. C. A.

31. Sang Tables (**Q20**, v. 2, p. 225).—There is a copy of *Sang's New Table of Seven-Place Logarithms*, 1915, in the Princeton University Library.

M. C. Shields

Fine Hall Library,
Princeton University

32. System of Linear Equations (**Q9**, v. 1, p. 203).—In this query it is noted that the method of Gauss and Seidel for solving a system of linear equations is not satisfactorily described in Whittaker & Robinson, *The Calculus of Observations* (London, 1924, and third ed., 1940, p. 255–256).

The difficulty arises from two errors by Whittaker & Robinson, (1) a failure to note that $m = n$ when giving the normal equations (we retain n below), and (2) an error in the definition of Q; this is stated (wrongly) to be the "sum of the squares of the residuals," while, in fact, the equations

$$a_{r1}x + a_{r2}y + \cdots + a_{rn}t - c_r = 0 \qquad r = 1 \text{ to } n$$

arise as conditions for minimizing the quantity

$$\begin{aligned} Q \equiv a_{11}x^2 + a_{22}y^2 + \cdots + a_{nn}t^2 \\ + 2a_{12}xy + 2a_{13}xz + \cdots - 2c_1x - 2c_2y - \cdots - 2c_nz + p. \end{aligned}$$

The method outlined by W. & R. is correctly based on the latter definition of Q.

The example given in **Q9** yields to the treatment outlined quite satisfactorily. It is

$$N_1 \equiv 2x + y - 1 = 0$$
$$N_2 \equiv x + 3y + 1 = 0,$$

with true solution $x = +4/5$, $y = -3/5$. Starting with values $x = \frac{1}{2}$, $y = -\frac{1}{3}$, as in **Q9**, we first evaluate N_1 and N_2, and then apply $\Delta x = -N_1/a_{11} = -\frac{1}{2}N_1$; re-evaluate N_2 (we shall have $N_1 = 0$) and apply $\Delta y = -N_2/a_{22} = -\frac{1}{3}N_2$; re-evaluate N_1, and put $x = -\frac{1}{2}N_1$ again, and so on. The values x, y, N_1, N_2, Q are given below:

Approx.	1st	Δx	2nd	Δy	3rd	Δx	4th	Δy	5th	Soln.
x	$+1/2$	$+1/6$	$+2/3$		$+2/3$	$+1/9$	$+7/9$		$+7/9$	$+4/5$
y	$-1/3$		$-1/3$	$-2/9$	$-5/9$		$-5/9$	$-1/27$	$-16/27$	$-3/5$
N_1	$-1/3$		0		$-2/9$		0		$-1/27$	0
N_2	$+1/2$		$+2/3$		0		$+1/9$		0	0
$Q-p$	$-7/6$		$-11/9$		$-37/27$		$-113/81$		$-340/243$	$-7/5$
	-1.17		-1.22		-1.37		-1.395		-1.3992	-1.40

As implied in **Q9**, the sum $N_1^2 + N_2^2$ shows an initial increase from $\frac{1}{9} + \frac{1}{4} = 13/36$ to $4/9 = 16/36$, but this is not relevant to the process.

J. C. P. Miller

CORRIGENDA

V. 2, p. 77, l. 32, *for* Steinmetz, *read* Steinitz; p. 342, l. 5, *for* 1856, *read* 1857.

Index to *MTAC* I: numbers 1-12, II: 13-20, 1943-1947 Supplementing the Index in number 12

In the 886 pages of these numbers a great mass of material has been accumulated during the past five years, and while in editing, cross-references have been used very freely, the problem of finding all material in the numbers dealing with a certain topic may still often be one involving the expenditure of much time. Indices bringing together references to all Articles, RMT, MTE, UMT, MAC, ACM, OAC, N, Q, QR, and Names, may often be useful, but others are evidently desirable. For example, when one is about to use a certain table in an important piece of work, one naturally wishes to know of its reliability. Eventually in MTE lists, for nos. 1–12 and 13–20, one may find all references to its errors in published discussions of *MTAC*. Now, however, we present also a single alphabetical author-index of all tables with listed errata, nos. 1–20. An index of this kind may well be of constant use. Readers are indebted to S.A.J. for conceiving and carrying through this valuable feature.

But a general Subject-Index to all the material published during the quinquennial period is also an obvious desideratum. Such an Index, which is presented herewith, was prepared during the past two years by Mrs. D. H. Lehmer, of Berkeley, California. The editors are profoundly grateful to her for this labor of love, providing a tool to render notable service for the inquirer interested in tabular material.

Here are the headings for the following Indices:

I, II: Numbers Issued

I: **1943**, **1**, Jan. (1–32); **2**, Apr. (33–68); **3**, July (69–100); **4**, Oct. (101–132); **1944**, **5**, Jan. (133–172); **6**, Apr. (173–204); **7**, July (205–308); **8**, Oct. (309–336); **1945**, **9**, Jan. (337–376); **10**, Apr. (377–408); **11**, July (409–432); **12**, Oct. (433–480). II: **1946**, **13**, Jan. (1–64); **14**, Apr. (65–96); **15**, July (97–148); **16**, Oct. (149–196); **1947**, **17**, Jan. (197–228); **18**, Apr. (229–288); **19**, July (289–320); **20**, Oct. (321–406).

II: Articles

II: Recent Mathematical Tables

Reviews by numbers, authors, pages

II: Corrigenda

II: Mathematical Tables—Errata

I, II: Mathematical Tables—Errata, Author Index

II: Unpublished Mathematical Tables

II, nos. 13–19: Mechanical Aids to Computation

II, no. 20: Automatic Computing Machinery

II, no. 20: Other Aids to Computation

II: Notes

II: Queries

II: Queries—Replies

I, II: Subject Index

II: Name Index

In spite of extensive research and many letters of inquiry there are still in the Index a few names without initials, among numerous names so appearing in the text.

CLAS

A. A
B. P
C. L
D. C
E. H

F. T

G. H

H. N
J. S

I. F
K. S

L. H
M. I

N. I
O. A

P. I

Q. A

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CLASSIFICATION OF TABLES, AND SUBCOMMITTEES

A. Arithmetical Tables. Mathematical Constants
B. Powers
C. Logarithms
D. Circular Functions
E. Hyperbolic and Exponential Functions
Professor DAVIS, *chairman*, Professor ELDER, Professor KETCHUM, Doctor LOWAN

F. Theory of Numbers
Professor LEHMER

G. Higher Algebra
Professor LEHMER

H. Numerical Solution of Equations
J. Summation of Series

I. Finite Differences. Interpolation
K. Statistics
Professor WILKS, *chairman*, Professor COCHRAN, Professor EISENHART, Professor FELLER, Professor HOEL

L. Higher Mathematical Functions
M. Integrals

N. Interest and Investment
O. Actuarial Science
Mister ELSTON, *chairman*, Mister THOMPSON, Mister WILLIAMSON

P. Engineering

Q. Astronomy
Doctor ECKERT, *chairman*, Doctor GOLDBERG, Miss KRAMPE

R. Geodesy
S. Physics, Geophysics, Crystallography
T. Chemistry
U. Navigation

V. Aerodynamics, Hydrodynamics, Ballistics

Z. Calculating Machines and Mechanical Computation
Professor CALDWELL, *chairman*, Doctor COMRIE, *vice-chairman*
Professor AIKEN, Professor LEHMER, Doctor MILLER, Doctor STIBITZ, Professor TRAVIS, Mister WOMERSLEY

EDITORIAL AND OTHER NOTICES

The addresses of all contributors to each issue of *MTAC* are given in that issue, those of the Committee being on cover 2. The use of initials only indicates a member of the Executive Committee, or an editor.

Due to the enlargement of *MTAC* and publication of illustrations, **beginning with 1947 the subscription price for each *calendar year* is $4.00, *payable in advance;*** ordinary single numbers $1.25. Earlier ordinary single numbers each $1.00, and all numbers for each of the years 1943 to 1946 inclusive, $3.00. Special single number 7, *Guide to Tables of Bessel Functions*, $1.75, and number 12, $1.50. All payments are to be made to National Academy of Sciences, 2101 Constitution Avenue, Washington, D. C. No reductions are made to Libraries or to Booksellers. No sample copies are distributed, but detailed descriptive circulars will be sent upon application.

CONTENTS

OCTOBER 1947

LANCASTER PRESS, INC., LANCASTER, PA.

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